Nuclear Magnetic Resonance

Week 2 NMR: The Vector Model of Pulsed NMR Spectroscopy

Problem Sets

- The problem sets should be solved most efficiently jointly and interactively during the problem sessions by the groups. Attendance and active participation is therefore very important to save time and to improve learning.

- After the session, each group should finalise their answers and upload the finished version (one single agreed version per group).

- Once we have all the answers from all the groups, then we will post them on Moodle, together with any required corrections

- These are an integral part of the course material.

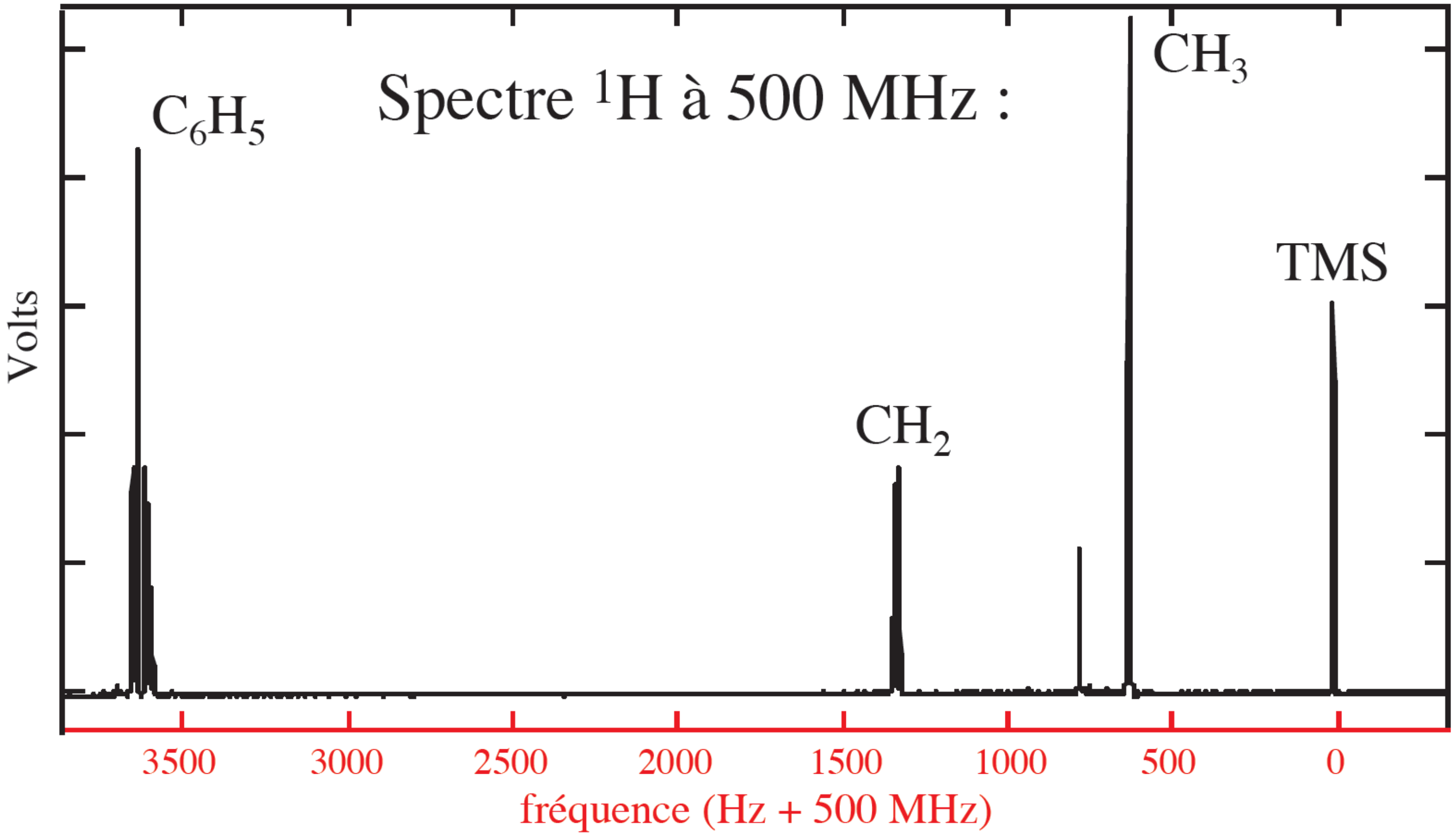
What we learned so far

- What an NMR spectrum looks like
- The principle interactions leading to NMR spectra:
- The Zeeman interaction leading to the Larmor frequency
- The chemical shift
- The scalar (J) coupling

Objectives

- Understand the vector model of time-domain NMR spectroscopy

The Zeeman Interaction



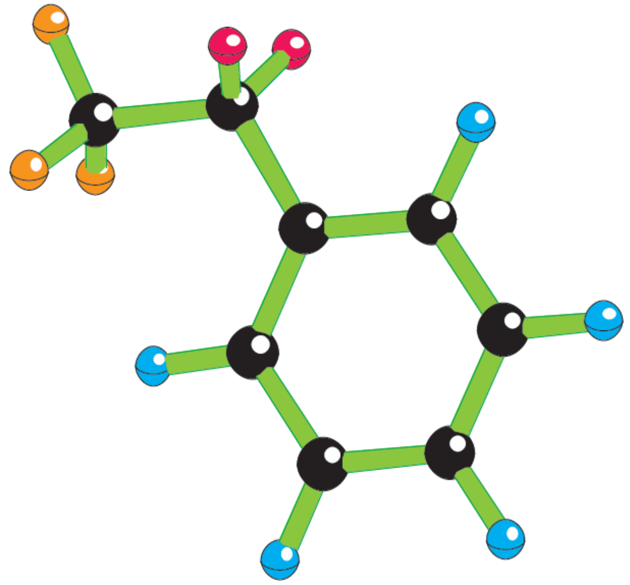
The Larmor Theorem states that the motion of a magnetic moment in a magnetic field (B_0) is a precession around that field at a frequency

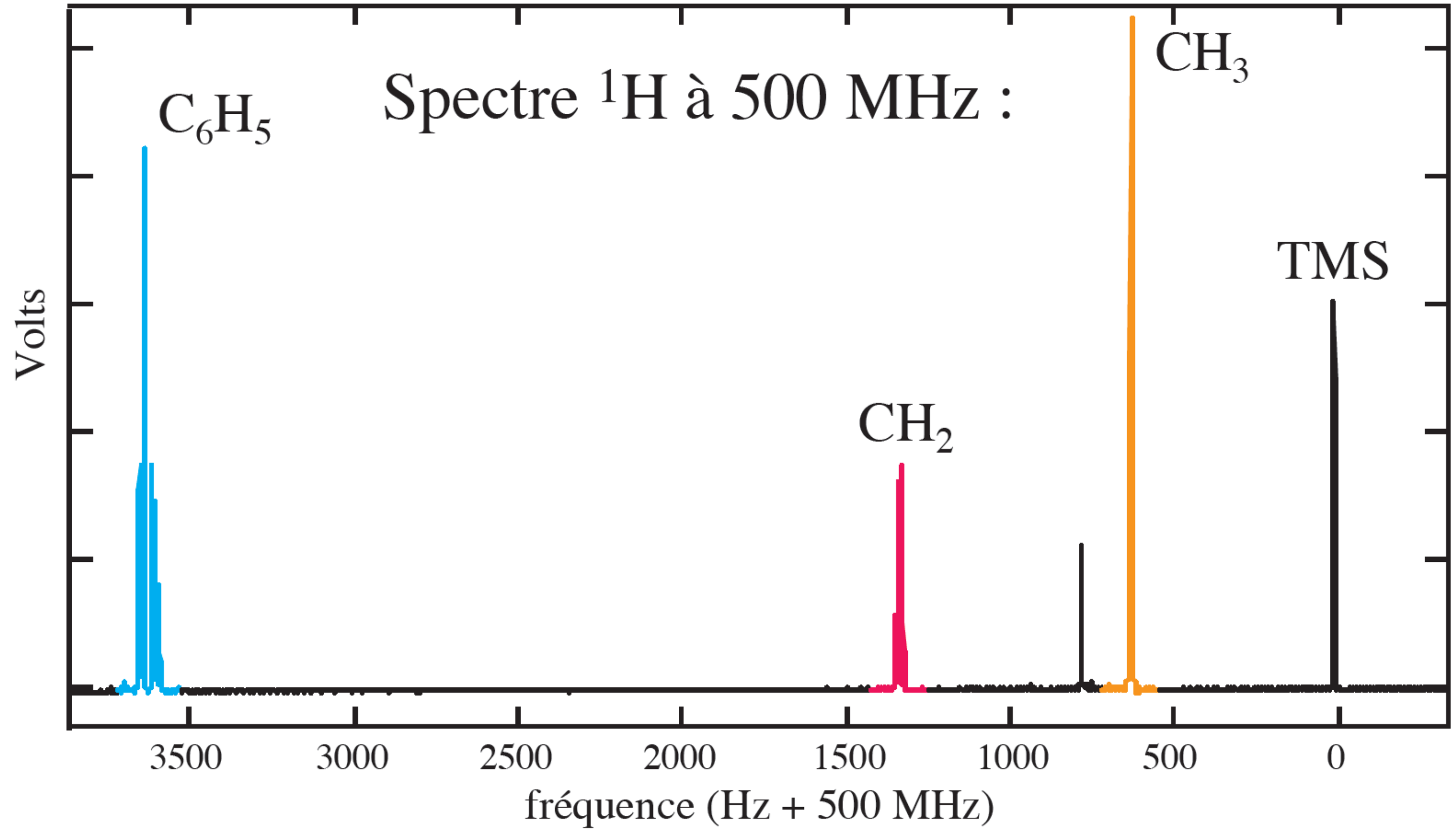
$$\boxed{\omega_0 = -\gamma B_0}$$

The Chemical Shift

Molécule :

$CH_3 - CH_2 - C_6H_5$

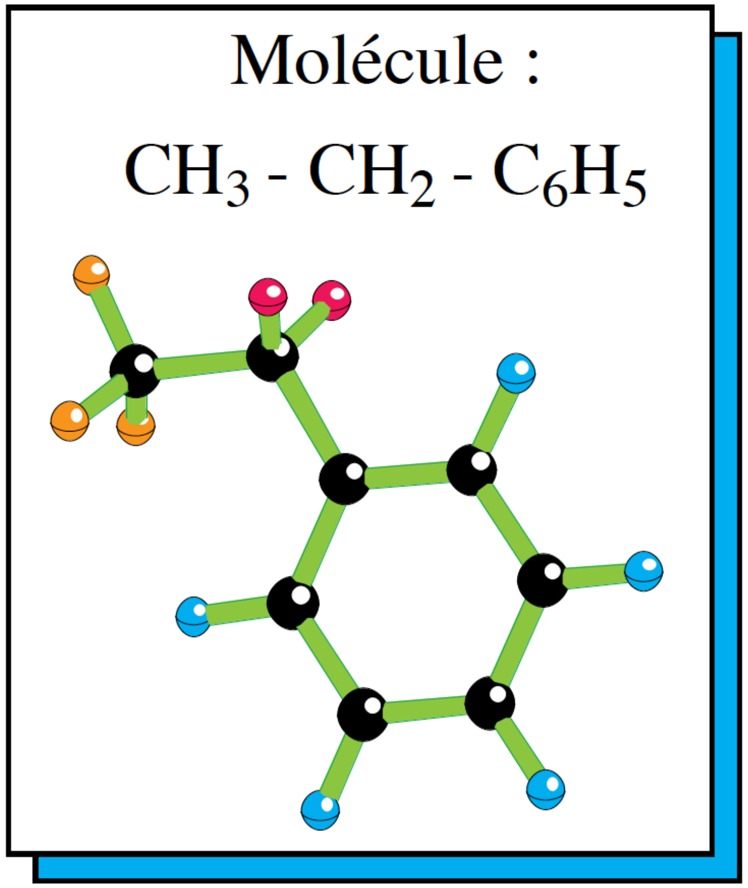




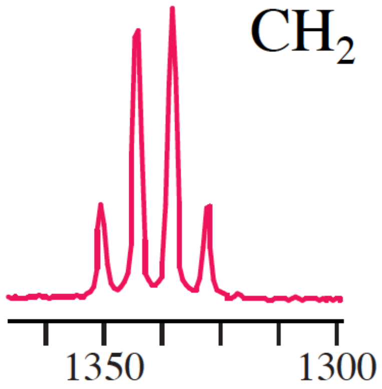
The Chemical Shift: the resonance frequency is modified very slightly (ppm) by the screening of the nucleus by the surrounding electrons.

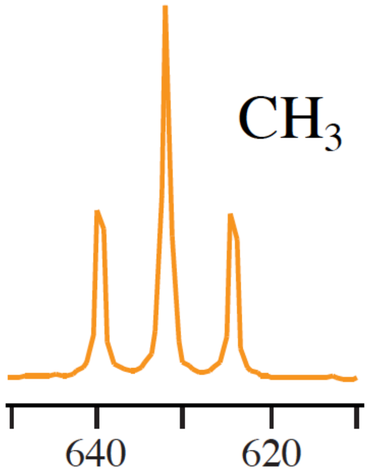
$$\boxed{\omega = -\gamma(1 - \sigma)B_0}$$

The Scalar (*J*) Coupling


Molécule :

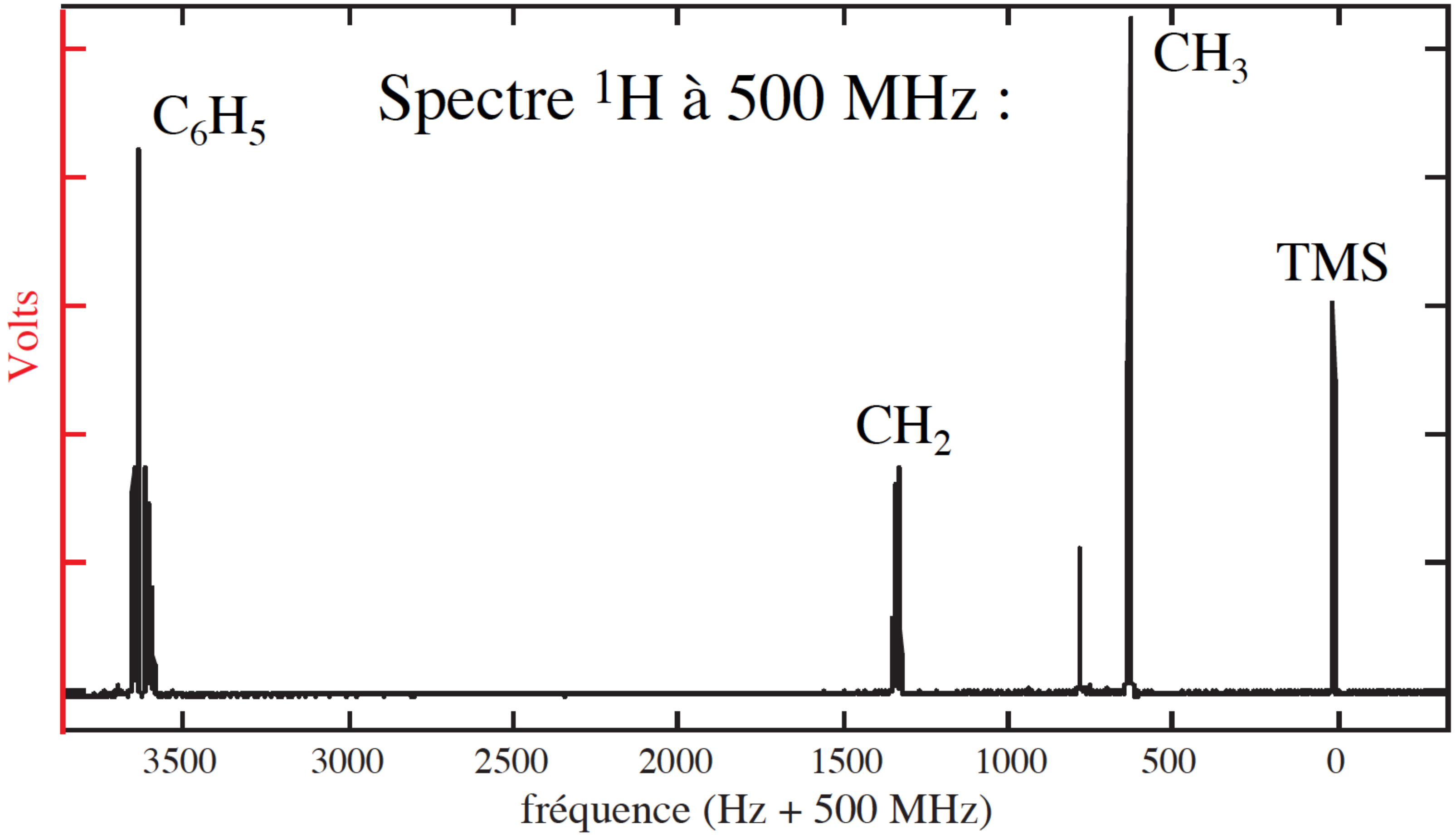
$CH_3 - CH_2 - C_6H_5$





The resonance frequency is also modified by the magnetic field perturbation caused indirectly *by the state of neighboring spins*. In solution the dominant effect is a through-bond Fermi contact interaction (or scalar interaction) referred to as *the J coupling*.

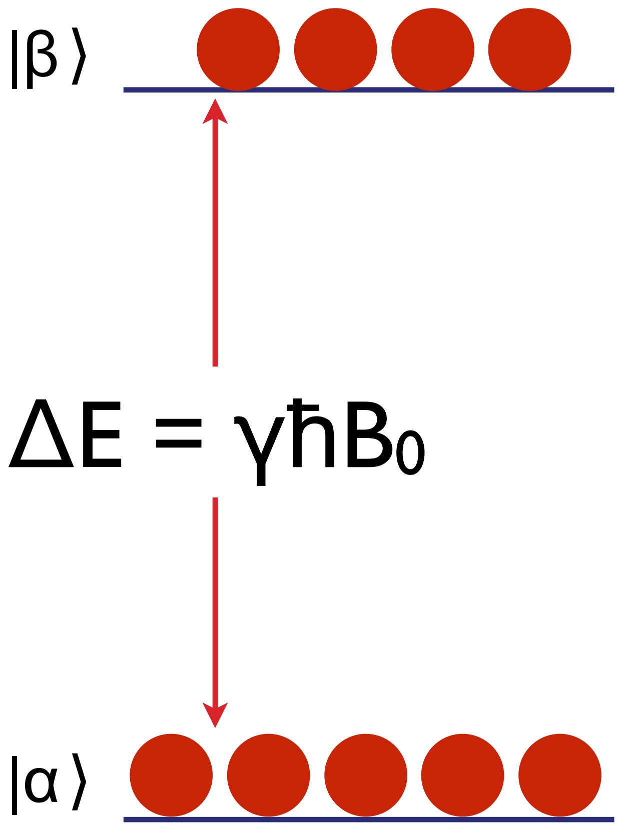
The y-axis?



The signal is measured in volts. The size of the signal is a complicated functions which depends on

1. Nuclear polarisation
2. The detection method
3. Geometry of the detection apparatus
4. Size of the sample (number of spins)

Nuclear Polarisation



The ratio of the populations of the α to β states, as seen last week, is

$$N_\alpha/N_\beta = \exp(-\Delta E/k_B T) \sim 1.0001$$

Nuclear polarisation, P, is defined as:

$$P = \tanh(\gamma\hbar B_0/2k_B T)$$

It varies from 0 for an unpolarised system, where the populations are equal, to 1 for a fully polarised system when only the ground state is populated.

The NMR signal is directly proportional to P.

At equilibrium, polarisation can be most obviously increased by increasing B_0, or by decreasing T. Both options are limited by practicalities in most cases. B_0^{max} (2021) ~ 23 T. T_{min} (solutions) ~ 270 K.

Methods to increase P beyond the equlibrium value are called “hyperpolarization” techniques. We will not discuss such methods here.

Geometry of the Detection Apparatus

A solenoid coil (left), a saddle coil (middle) and a Helmholtz coil (right). The three of them yield a homogeneous magnetic field in the center with a direction represented by an arrow.
(from A.J. Perez-Linde, PhD thesis, Nottingham)

A shoulder coil for MRI. Coil geometry needs to be compliant with many factors, including the sample! Solenoids are often not possible.

The ideal cylindrical geometry is a cylinder of infinite length. In practice, the best known approximation of this configuration is the solenoid coil. This coil gives the best sensitivity of magnetic resonance among all resonator designs.

In comparison a saddle coil is roughly a factor √2 less sensitive. However, most solution-state NMR is done using saddle coils, because they provide access for the sample from the top. Coil geometry needs to be compliant with many factors, including the sample! Solenoids are often not possible.

Further reading: Homogeneous resonators for magnetic resonance: A review
https://www.sciencedirect.com/science/article/pii/S1631074807002652

Size and Concentration of the Sample

Sample Size: If all other factors are equal, the NMR signal is proportional to the number of spins. Doubling the volume of the sample inside a given coil will thus double the signal. However, doubling the volume in a properly optimised system will involve also doubling the coil volume, which will also increase the noise.

Sample Concentration: In a given sample, the NMR signal will be directly proportional to the number of spins. Thus:

S ∝ [concentration].

e.g. Doubling the concentration of a sample will double the signal size. This also means that **relative signal intensities within an NMR spectrum are quantitative reporters of relative concentrations**. (As long as the experiment is carefully done to avoid some pitfalls).

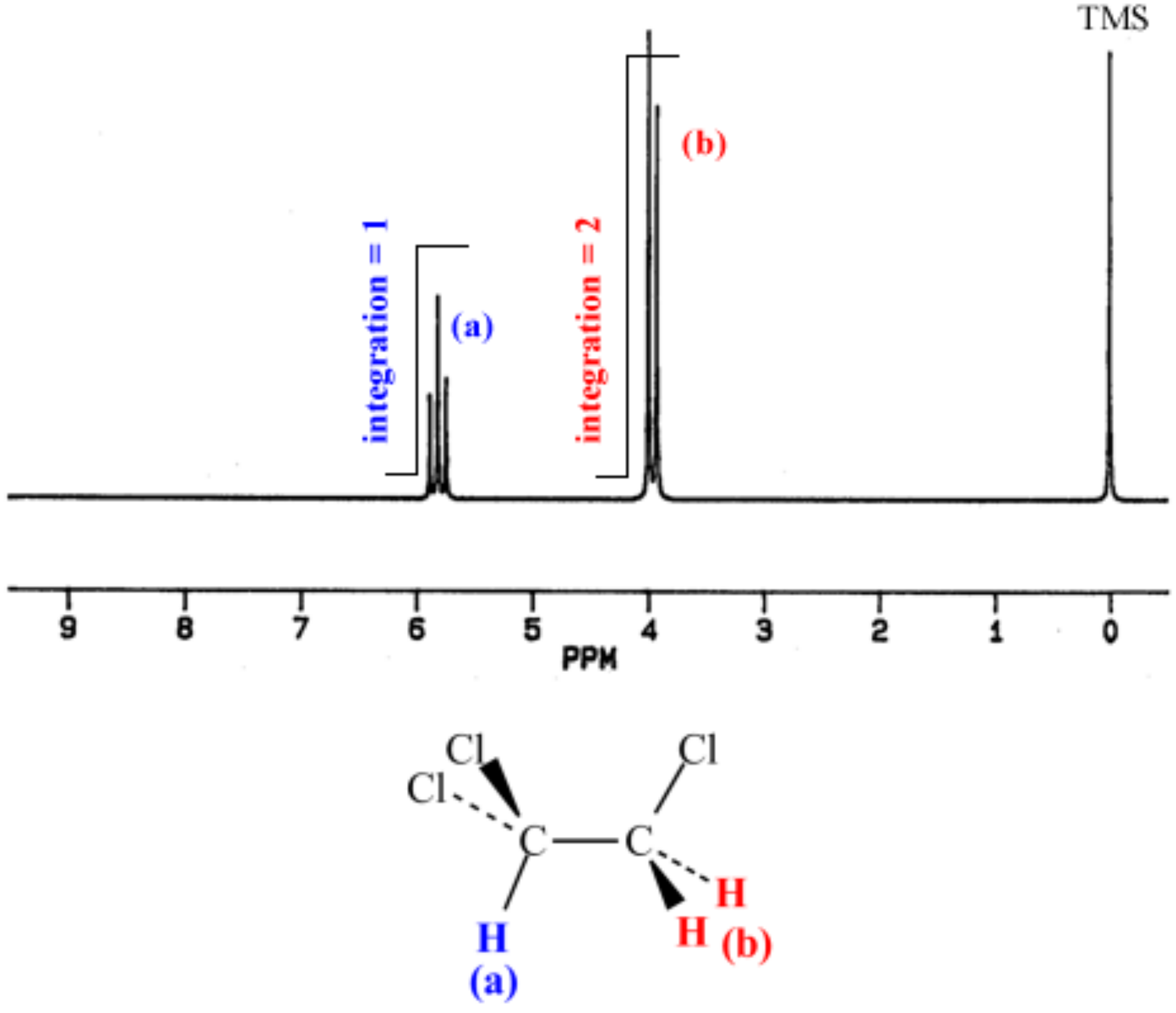


figure: https://scilearn.sydney.edu.au/organicspectroscopy/?type=nmr&page=the%20spectrum

Summary

The signal *S* detected in an NMR experiment based on the inductive coupling of the spins with a pickup coil can be expressed by the classic formula of Hoult and Richards:

$$S \propto \frac{B_1}{I} V_s N \gamma \hbar^2 S(S+1) \frac{\omega_0^2}{3kT\sqrt{2}}$$

where, V_s is the sample volume, *S* is the spin angular momentum quantum number, *N* is the spin density (number of spins per unit volume). The factor B_1/I, the magnetic field per unit current, is defined as the coil sensitivity, and is inversely proportional to the diameter of the coil. The NMR signal-to-noise ratio is derived by dividing this signal by a noise voltage V_{noise},

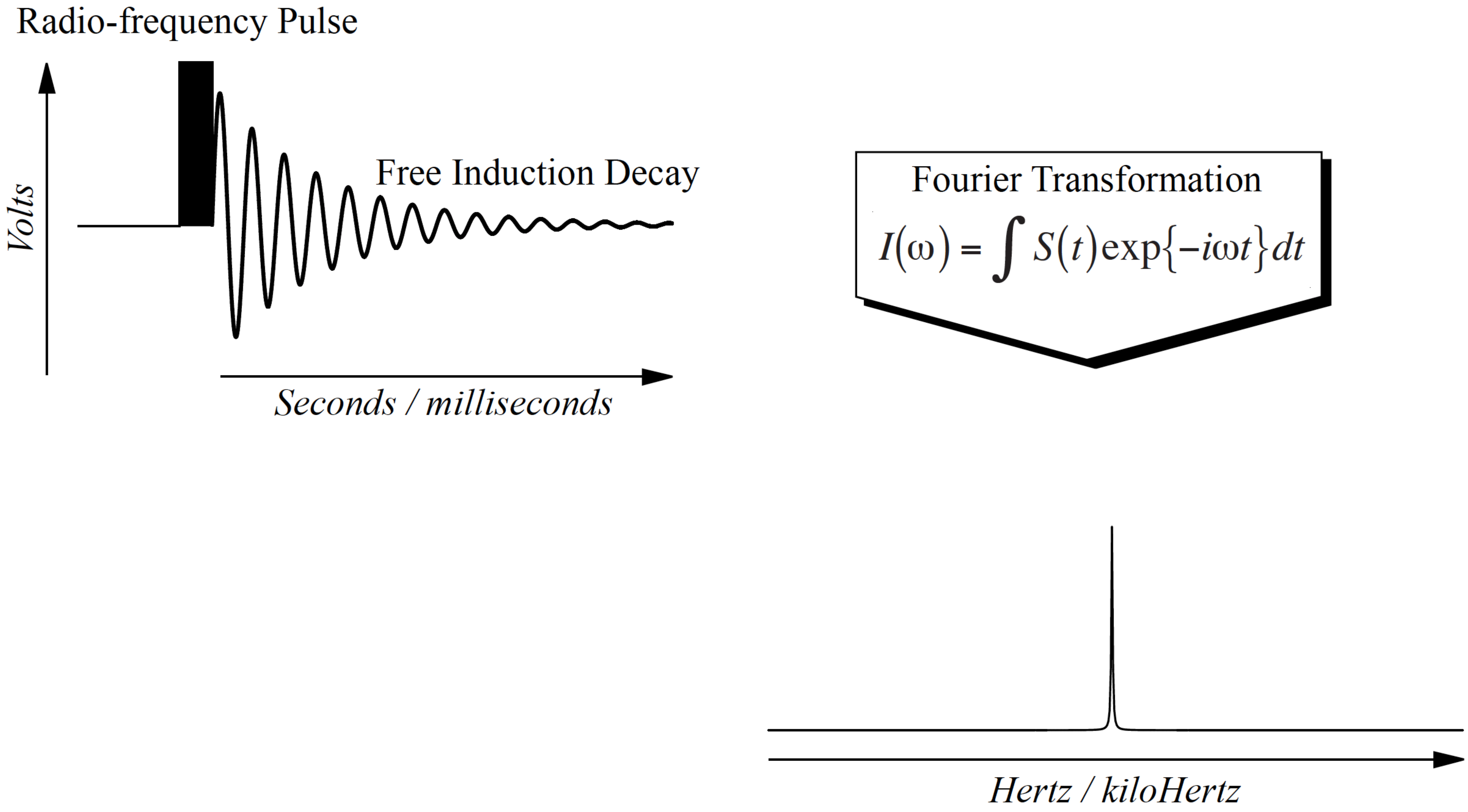
$$V_{noise} \propto \sqrt{4k(\Delta f)(T_N)}$$

where Δf is the receiver bandwidth, and T_N is the noise temperature which is a function of both the temperature and the effective resistance of the sample and the coil:

$$T_N \propto \frac{T_{coil} R_{coil} + T_{sample} R_{sample}}{R_{coil} + R_{sample}}$$

D. J. Hoult, R. E. Richards, J. Magn. Reson. 1976, 24, 71– 85
from https://doi.org/10.1002/anie.201410653

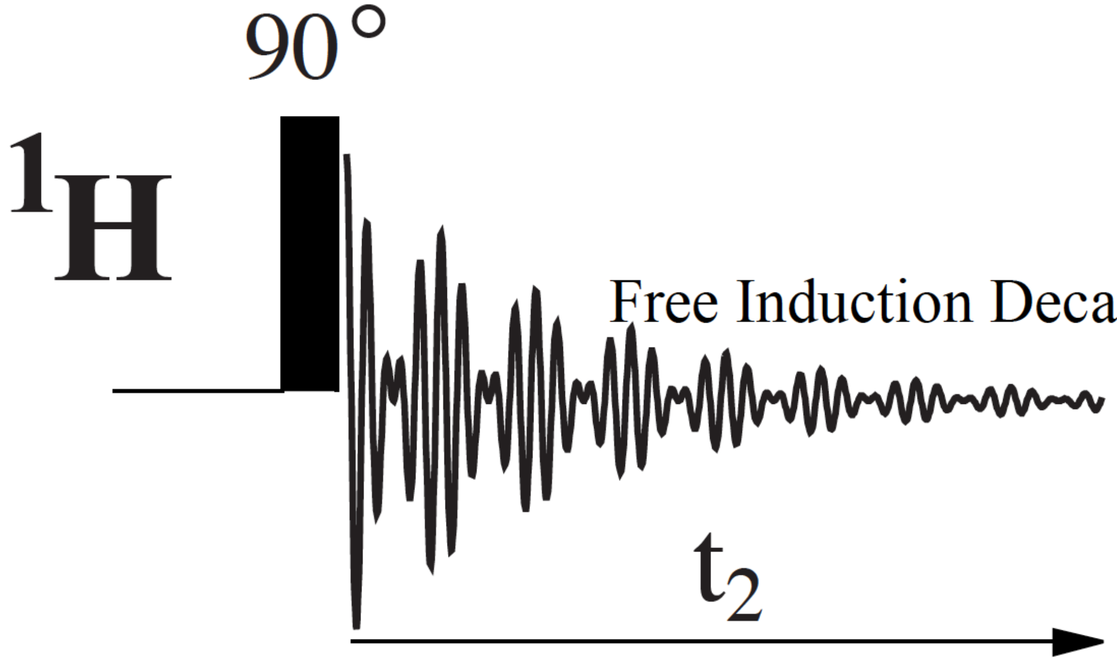
Pulsed Fourier Transform NMR



Fourier Transformation

$$I(\omega) = \int S(t)\exp\{-i\omega t\}dt$$

Pulsed Fourier Transform NMR



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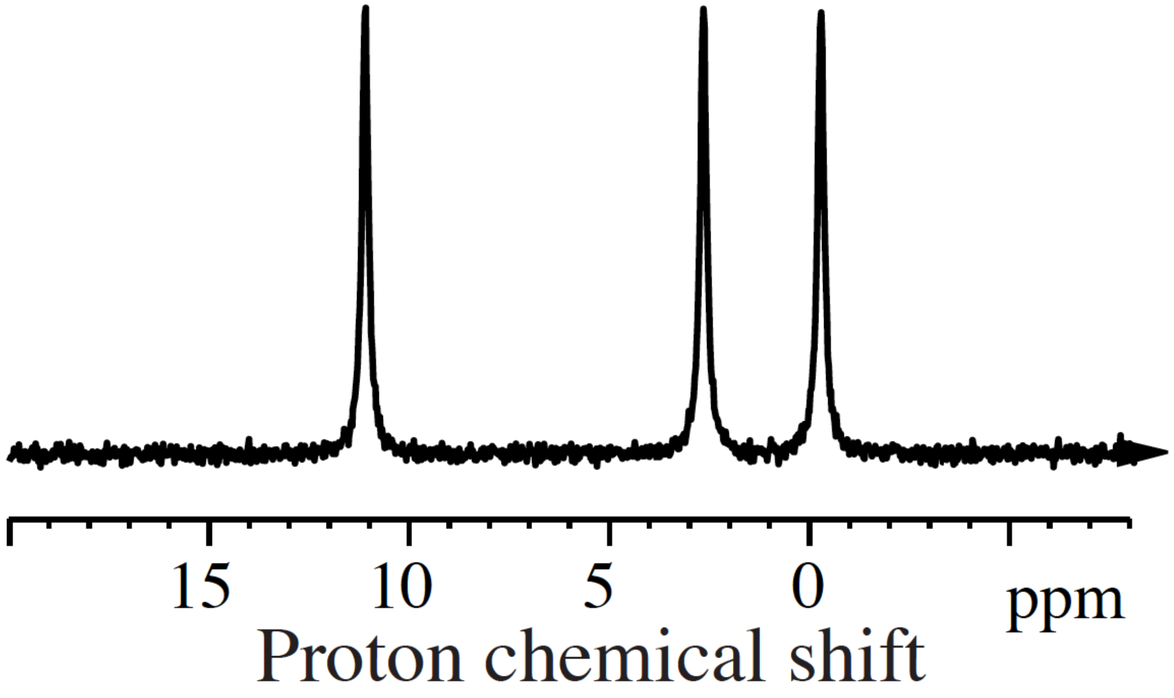


Pulsed Fourier Transform NMR

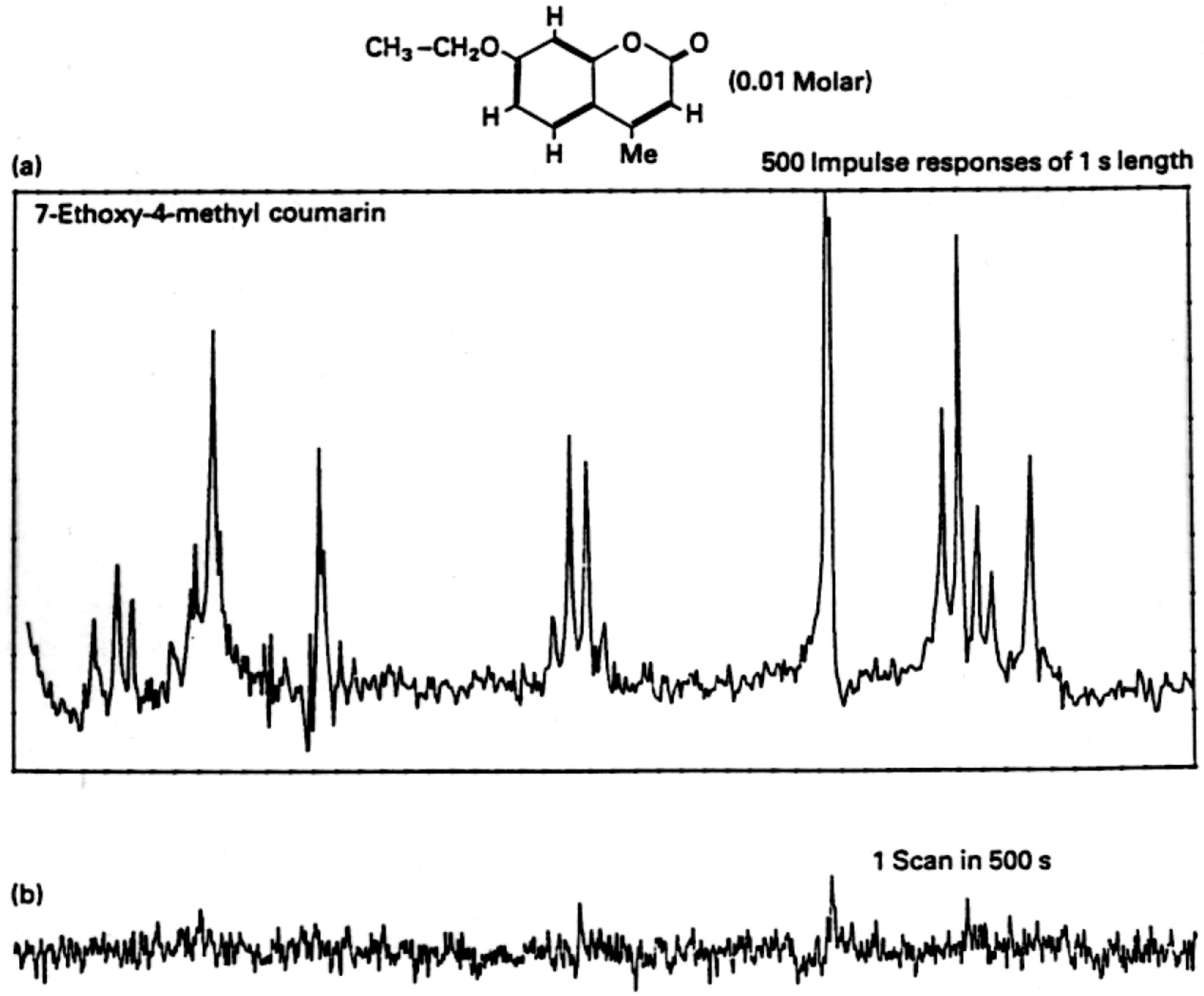
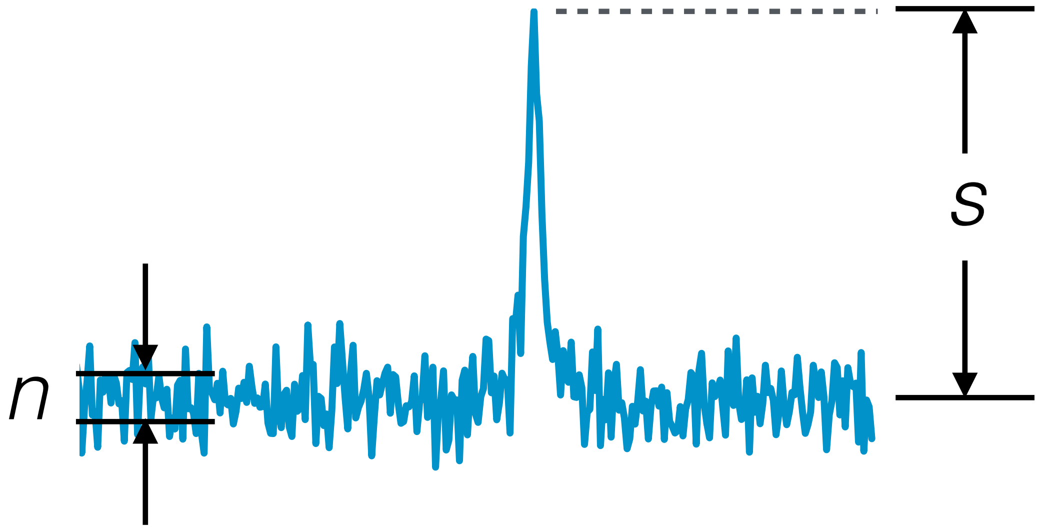


FIG. 4.3.4. 60-MHz proton magnetic resonance spectra of 7-ethoxy-4-methyl coumarin. (a) Fourier transform of 500 free induction signals recorded in 500 s. (b) Single scan recorded in 500 s by slow passage on the same instrument. (Reproduced from Ref. 4.130.)

from Ernst, Bodenhausen, Wokaun

NMR Sensitivity: the signal-to-noise ratio

Because the population difference between the energy levels is very small, the NMR signal tends to be weak. As a result it is almost never the case that the spectrum from a single acquisition has sufficient a **signal-to-noise** ratio to be useful.



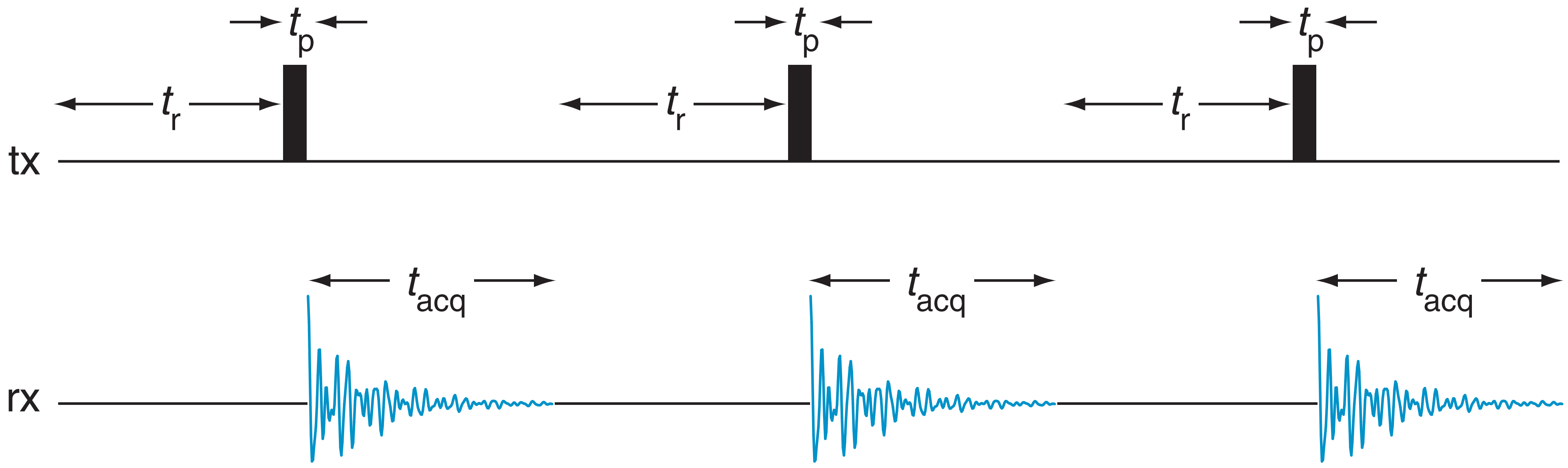
$$\text{SNR} = \frac{\textit{signal intensity}}{\textit{rms noise}} = \frac{s}{n}$$

In order to improve the signal-to-noise ratio we use time averaging. We repeat the experiment a number of times and then add together the resulting spectra. The signal part adds up so that after N experiments the signal will be N times stronger. However, the noise, because it is random, adds up more slowly typically increasing as $\sqrt{N}$. (Note that adding random noise does not lead to it cancelling out).

Repeating the experiment N times thus yields an improvement in the signal-to-noise ratio by a factor of $\sqrt{N}$.

see Keeler, chapter 2 and chapter 5

Time averaging to improve SNR

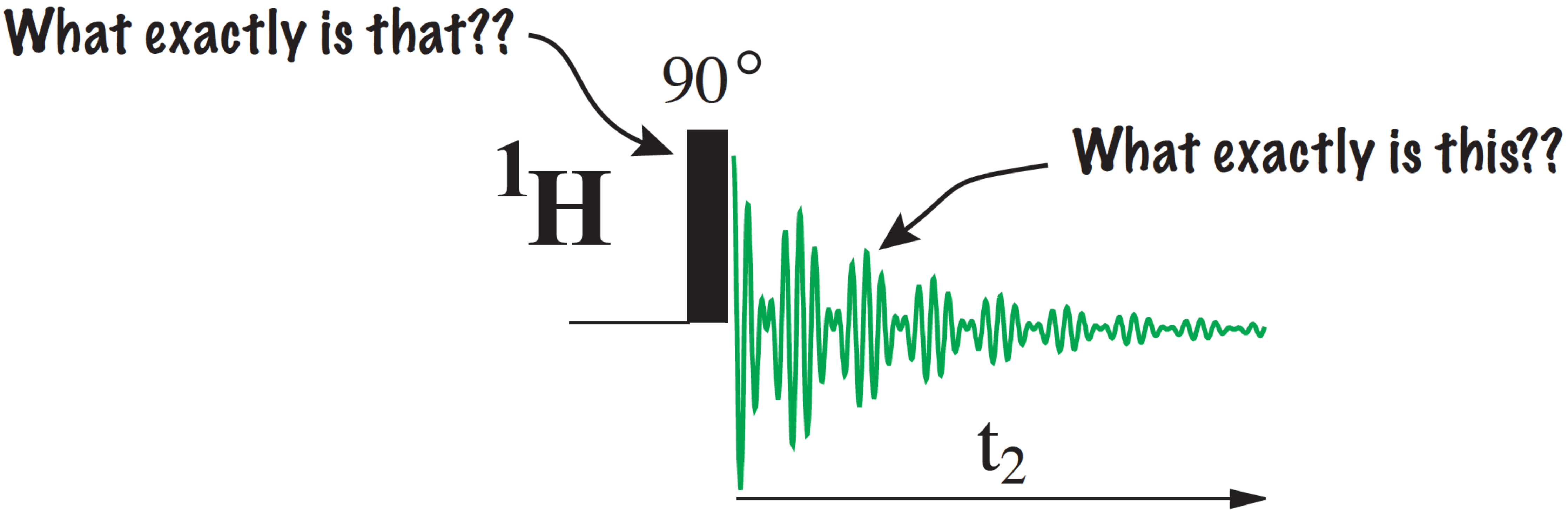


Timing diagram showing how a basic NMR spectrum is recorded. The line marked ‘tx’ shows the location of “pulses.” The NMR signal is detected by a receiver during the times shown on the line marked rx. During time t_r the spins are allowed to return to equilibrium.

A very short “pulse” is applied for time t_p and then the resulting signal or “free indiction decay” (FID) is recorded for time t_{acq}. In order to improve the signal-to-noise ratio, the whole process is repeated several times over and the FIDs are added together; this process is called time averaging. Here, the experiment is repeated three times.

see Keeler, chapter 2 and chapter 5

Pulsed Fourier Transform NMR



What exactly is that??

What exactly is this??

NMR experiments observe the voltage induced in a detection coil by *oscillating bulk magnetic moments* as they return to equlibrium.

The oscillations are induced by *perturbing* the system with a radio-frequency pulse. (Since the system is time-dependent, it must be out of equilibrium)

In following we will investigate the origin of these dynamic nuclear magnetic moments.

Understanding NMR Spectroscopy

We made quite a lot of progress towards understanding the form of NMR spectra by working from the energy levels and selection rules.

However, this has brought us no closer to understanding how even the simplest pulse acquire NMR experiment actually works.

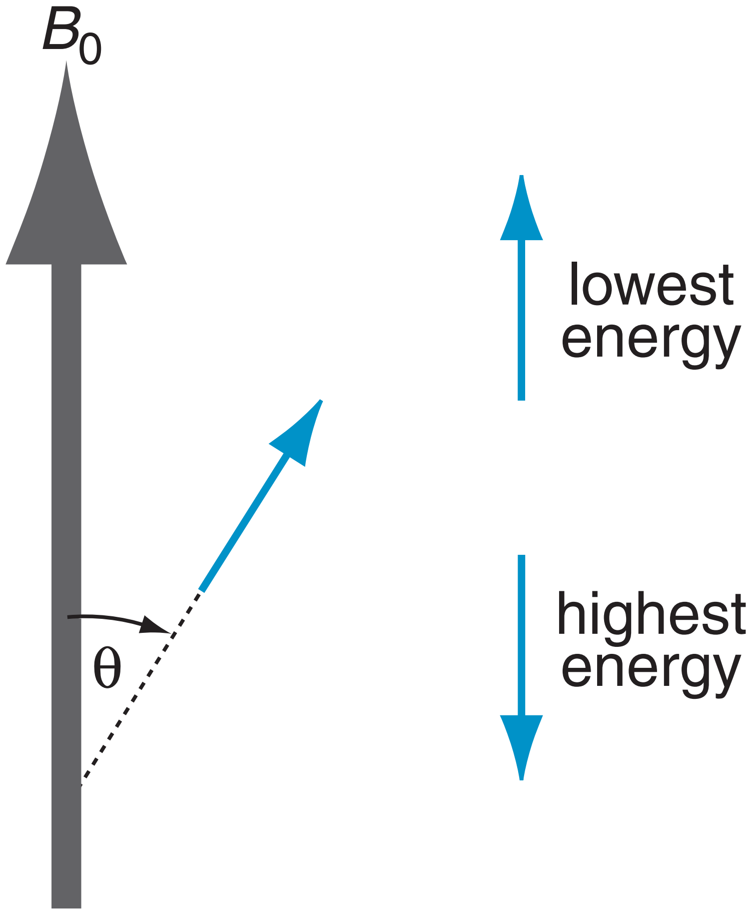
Ultimately it is only quantum mechanics which will give us the complete understanding we are looking for.

However, before we embark on the full rigours of that approach we will spend some time exploring the much simpler **vector model**.

Nuclear Magnetic Moments

Some nuclei posses spin angular momentum. It turns out that associated with this angular momentum there is always a ***nuclear spin magnetic moment***; what this means is that the nucleus generates a small magnetic field (as if it were a tiny bar magnet):

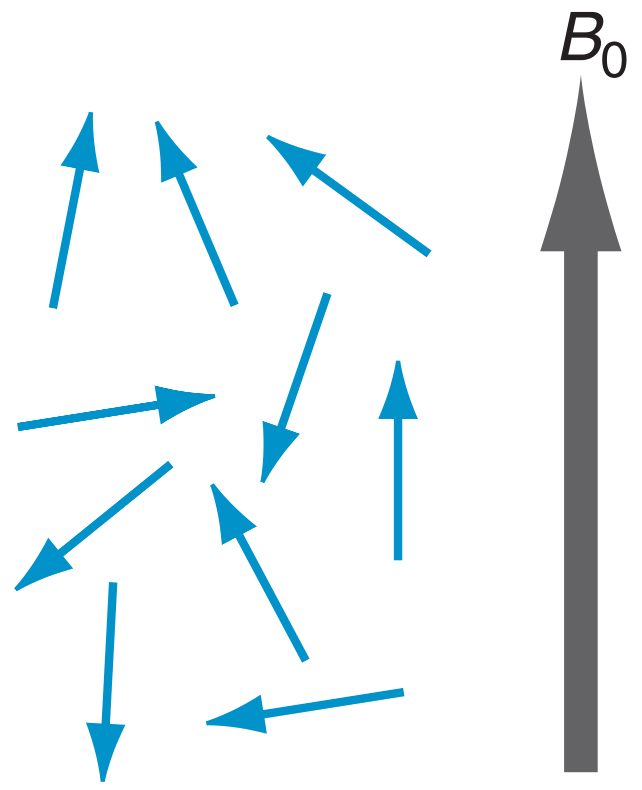
$$\mu = \gamma h I$$



The energy of interaction between a magnetic moment and an applied magnetic field depends on the angle θ between the magnetic moment and the field direction.

The lowest energy arrangement is when the magnetic moment is parallel to the field (θ = 0), and the highest energy arrangement is when the moment is anti-parallel to the field (θ = π radians).

Bulk Magnetization of the Sample

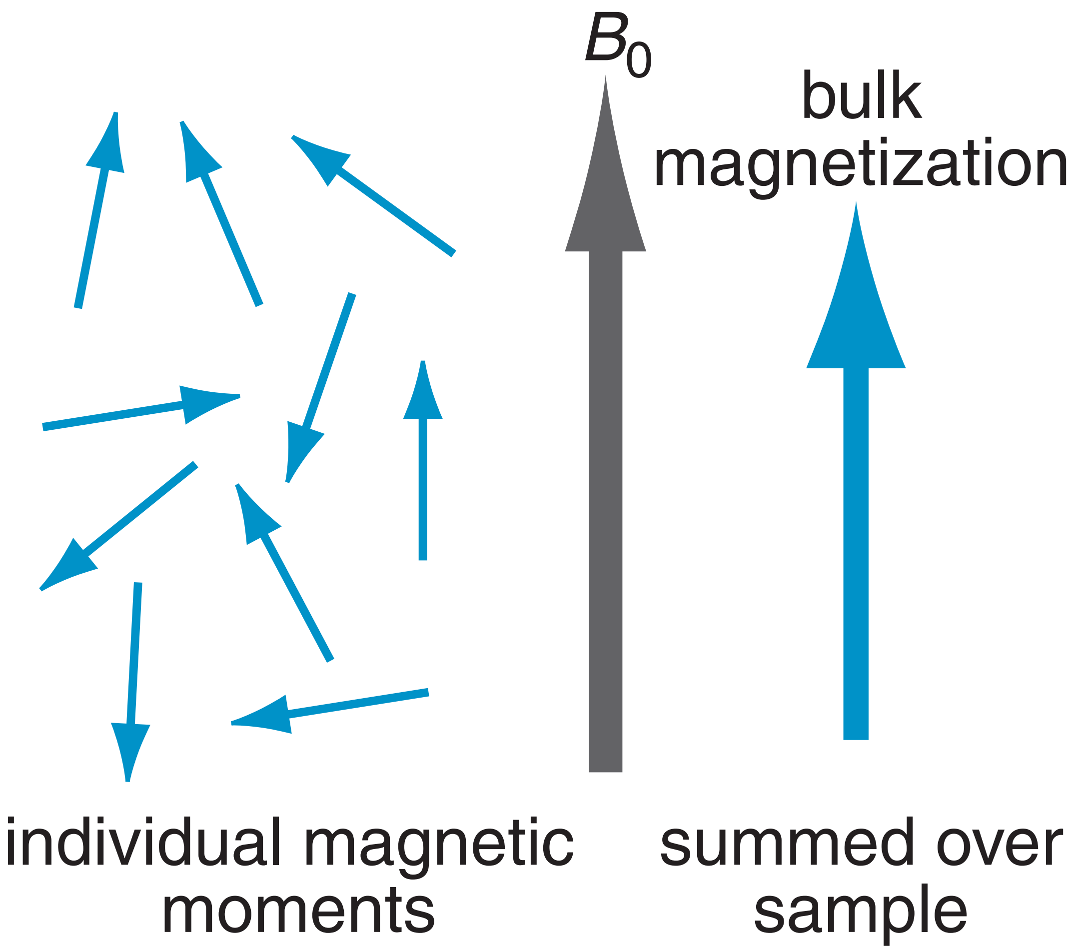


individual magnetic moments

As a result of the disruption due to thermal motion, the individual magnetic moments are not all able to adopt the lowest energy arrangement in which they align with the field.

For nuclear magnetic moments the interaction with the field is so small that, across the sample, the arrangement of the moments is almost random.

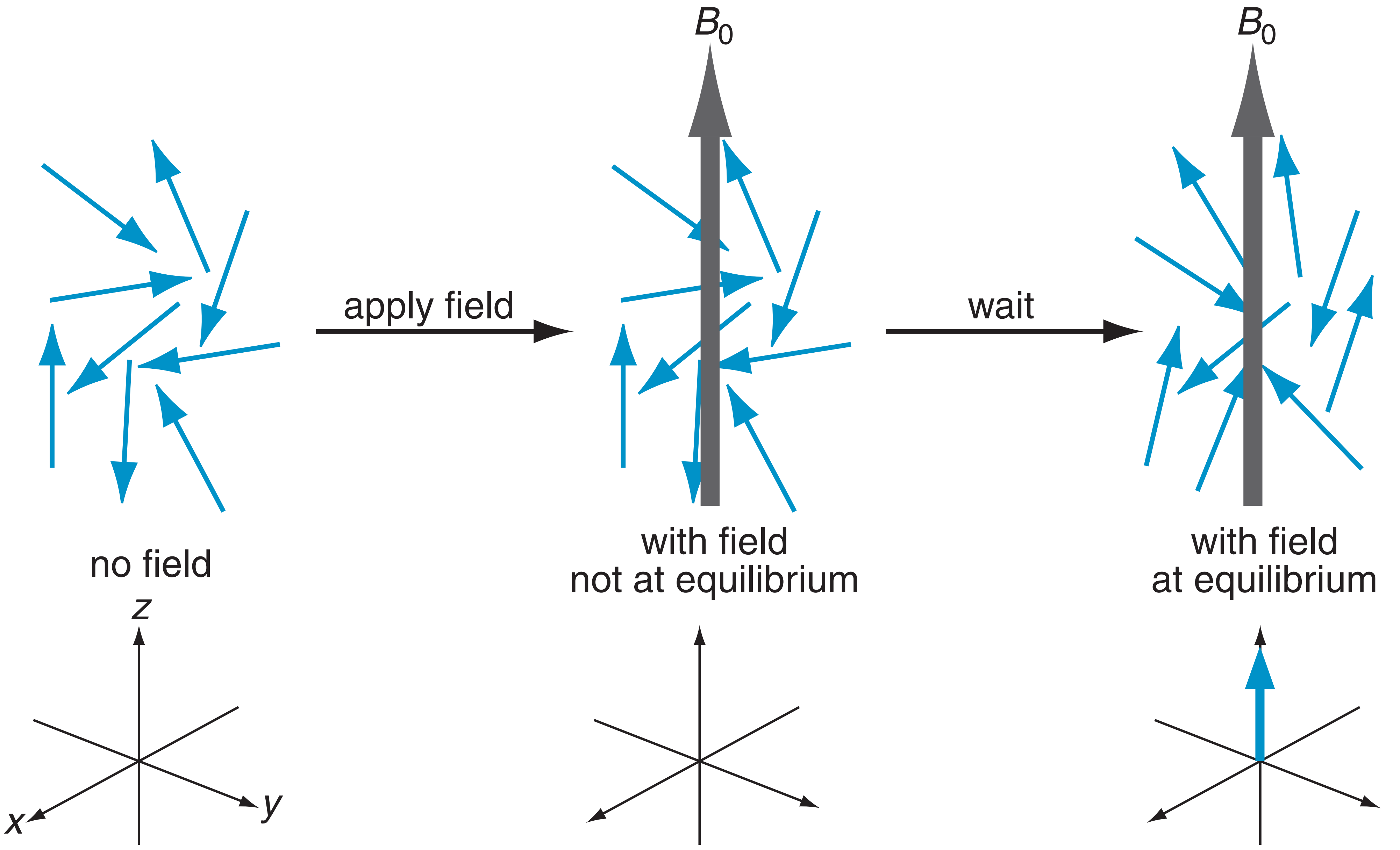
Bulk Magnetization of the Sample



However, there is a small (<0.1%) preference for alignment with the field and this, when averaged over the sample, gives rise to a bulk magnetization, **M**, of the sample, parallel to the field direction.

This magnetization can be represented by a vector, called the bulk magnetization vector.

Nuclear Magnetic Relaxation to Equilibrium

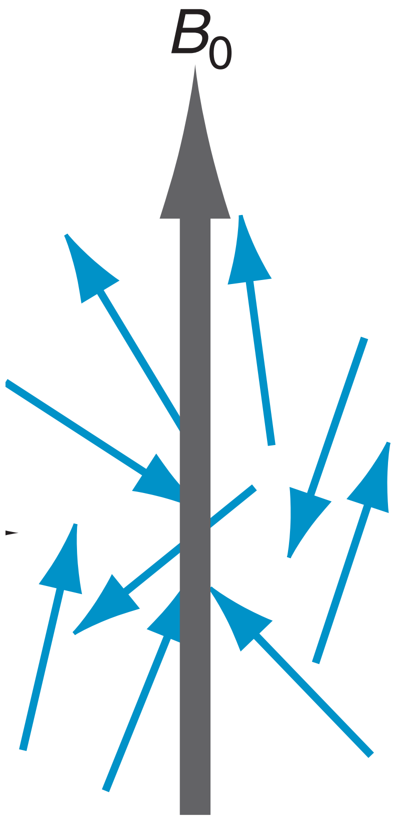


In the absence of a magnetic field all orientations have the same energy: moments are oriented randomly.

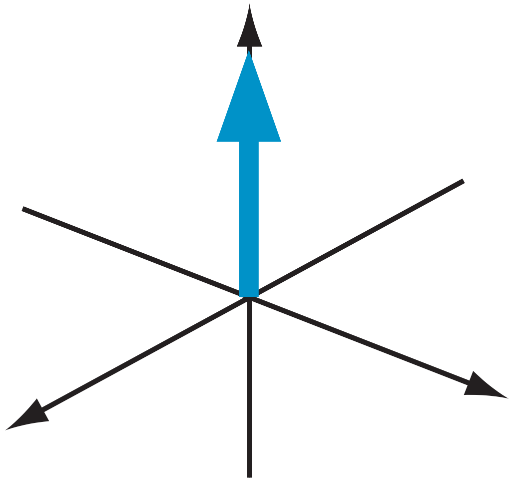
in $\mathbf{B}_0$ to start with the magnetic moments are still oriented randomly so there is no net magnetization

Over time random molecular motion ensures that the lower-energy orientations are preferentially populated

Equilibrium

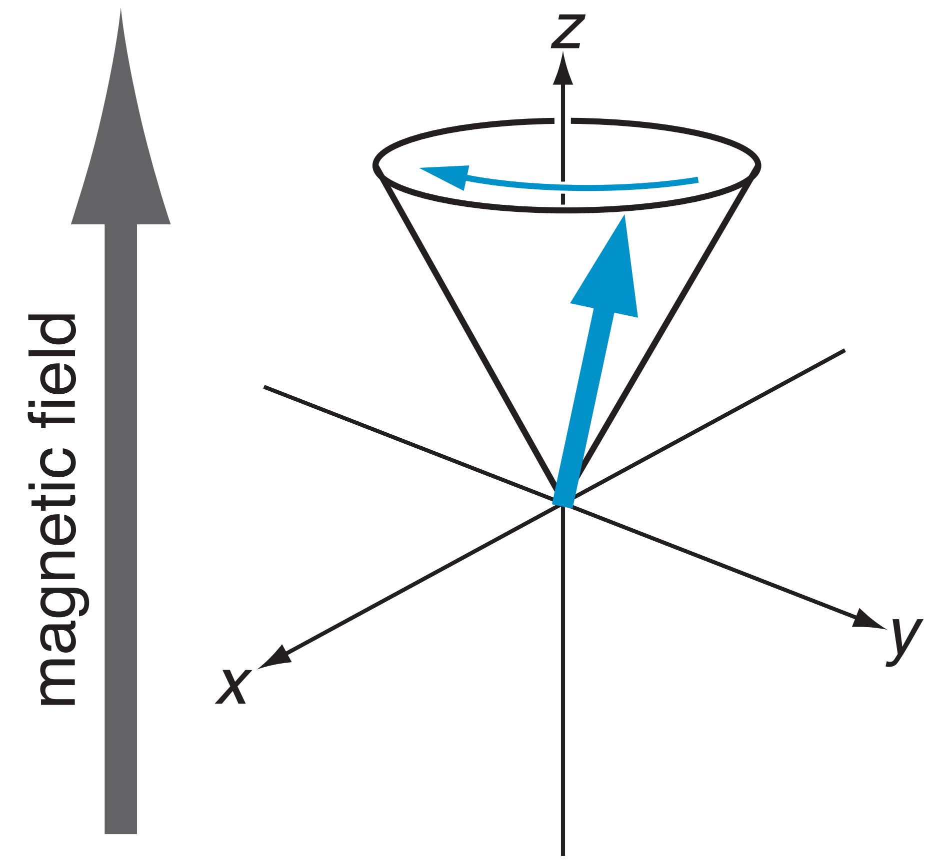


with field
at equilibrium



If we choose (by convention) the applied magnetic field $\mathbf{B}_0$ to be aligned along the z direction of the coordinate system, the energy of the moments is independent of the orientation of the moment in the xy-plane. As a result there is no energetic preference for any particular orientation in the xy-plane, so at equilibrium we expect that the x- and y-components of the individual magnetic moments will be distributed randomly. At equilibrium **M** is aligned with the direction of the magnetic field $\mathbf{B}_0$

Larmor Precession



$\omega_0 = -\gamma B_0$ in rad s^{-1}

or

$\upsilon_0 = -\gamma B_0 / 2\pi$ in Hz

At equilibrium the magnetisation vector is stationary.
However, **if** the magnetization vector was tipped away from the z-axis, by an angle β then the magnetization vector would rotate about the direction of the magnetic field sweeping out a cone with angle β.
This kind of motion is called **precession** – the vector is said to precess about the field.

The Larmor Theorem (Proof)

The Larmor Theorem (Proof). The theorem states that the motion of the magnetic moment $\boldsymbol{M}$ in a magnetic field $\boldsymbol{B}$ is a precession around that field. The proof is as follows:

(i) the energy of the moment in the field is

$$E = -\boldsymbol{M} \cdot \boldsymbol{B}$$

(ii) as a consequence the moment experiences a torque

$$\boldsymbol{C} = \boldsymbol{M} \times \boldsymbol{B}$$

(iii) the torque is equal to the time derivative of the angular momentum

$$\boldsymbol{C} = \frac{d\hbar \boldsymbol{I}}{dt} = \boldsymbol{M} \times \boldsymbol{B}$$

multiply by γ

$$\frac{d}{dt}\boldsymbol{M} = \gamma \boldsymbol{M} \times \boldsymbol{B}$$

To show this let us use a frame of reference rotating w.r.t the fixed axes with an angular velocity represented by the vector w. Classical mechanics tells us that the evolution rates of M as viewed in the fixed (lab) frame are related by

$$\left(\frac{d}{dt}\boldsymbol{M}\right)_{rot} = \left(\frac{d}{dt}\boldsymbol{M}\right)_{lab} + \boldsymbol{M} \times \boldsymbol{\omega} = \boldsymbol{M} \times (\gamma \boldsymbol{B} + \boldsymbol{\omega})$$

We choose $\boldsymbol{\omega} = -\gamma \boldsymbol{B}$, so that the effective field vanishes, and $\boldsymbol{M}$ is time independent in the rotating frame. From a lab point of view it is therefore rotating around $\boldsymbol{B}$ with a rotation vector $\boldsymbol{\omega} = -\gamma \boldsymbol{B}$. In a fixed field B_0, the rotation (or precession) frequency

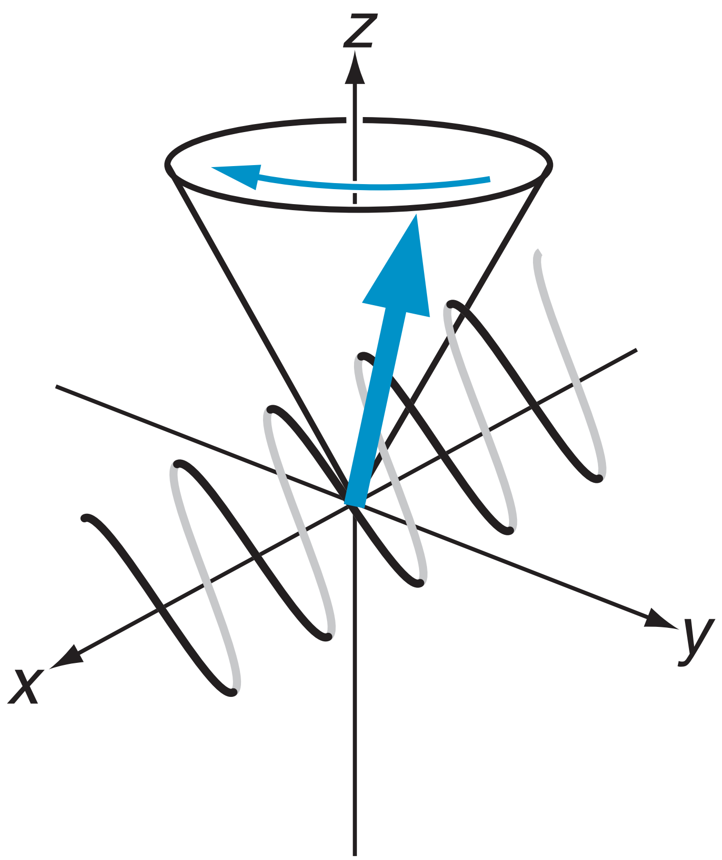
$\omega_0 = -\gamma B_0$.

is called the Larmor frequency.

Precession (Callaghan)

https://www.youtube.com/watch?v=7aRKAXD4dAg

The NMR Signal



$\omega_0 = -\gamma B_0$ in rad s^{-1}

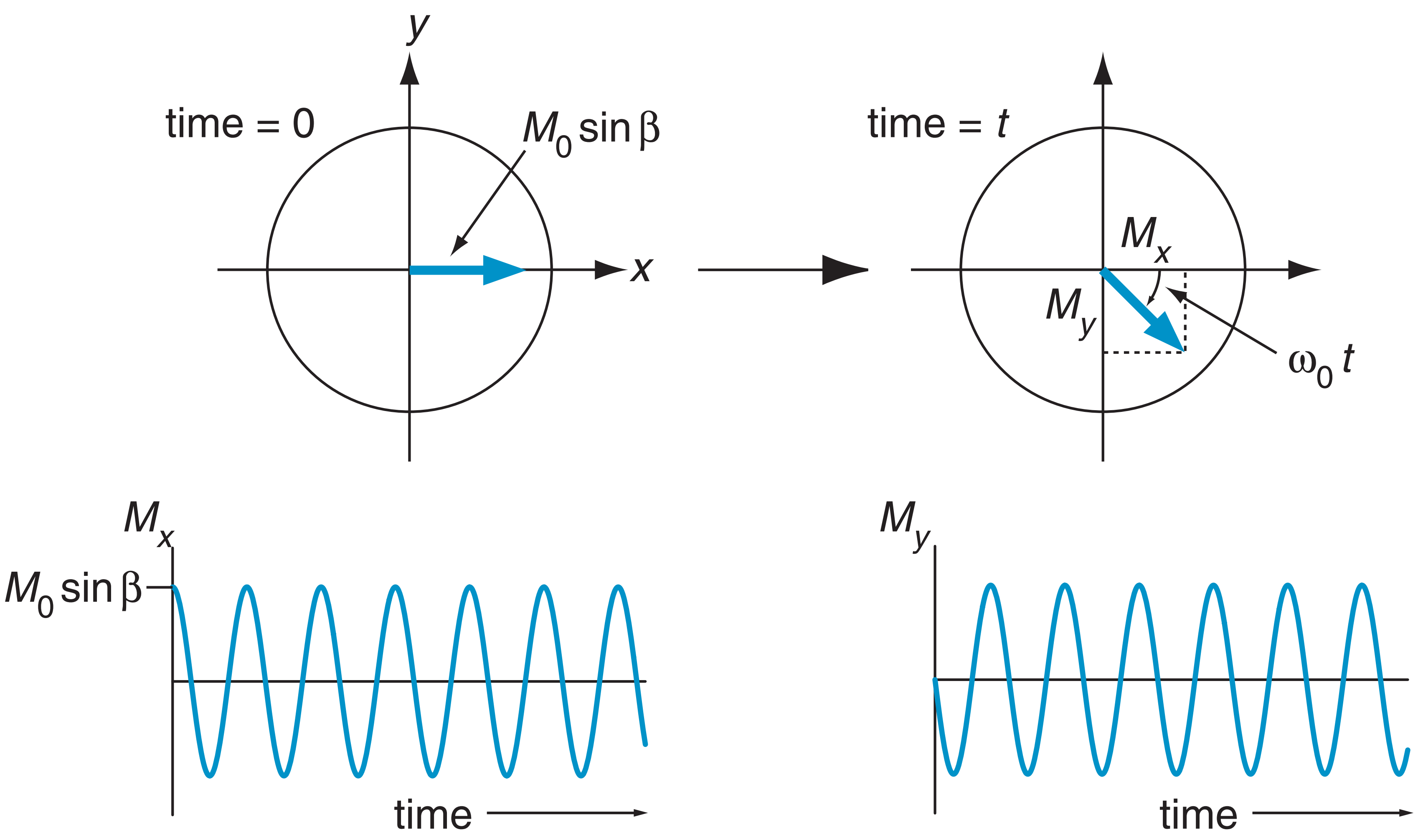
or

$\upsilon_0 = -\gamma B_0/2\pi$ in Hz

The precession of the magnetization vector is what we actually detect in a pulsed NMR experiment. All we have to do is to mount a small coil of wire round the sample, with the axis of the coil aligned in the xy-plane.

The precessing magnetization induces a current in the coil which we can amplify and then record – this is the free induction signal or, more usually, free induction decay (FID).

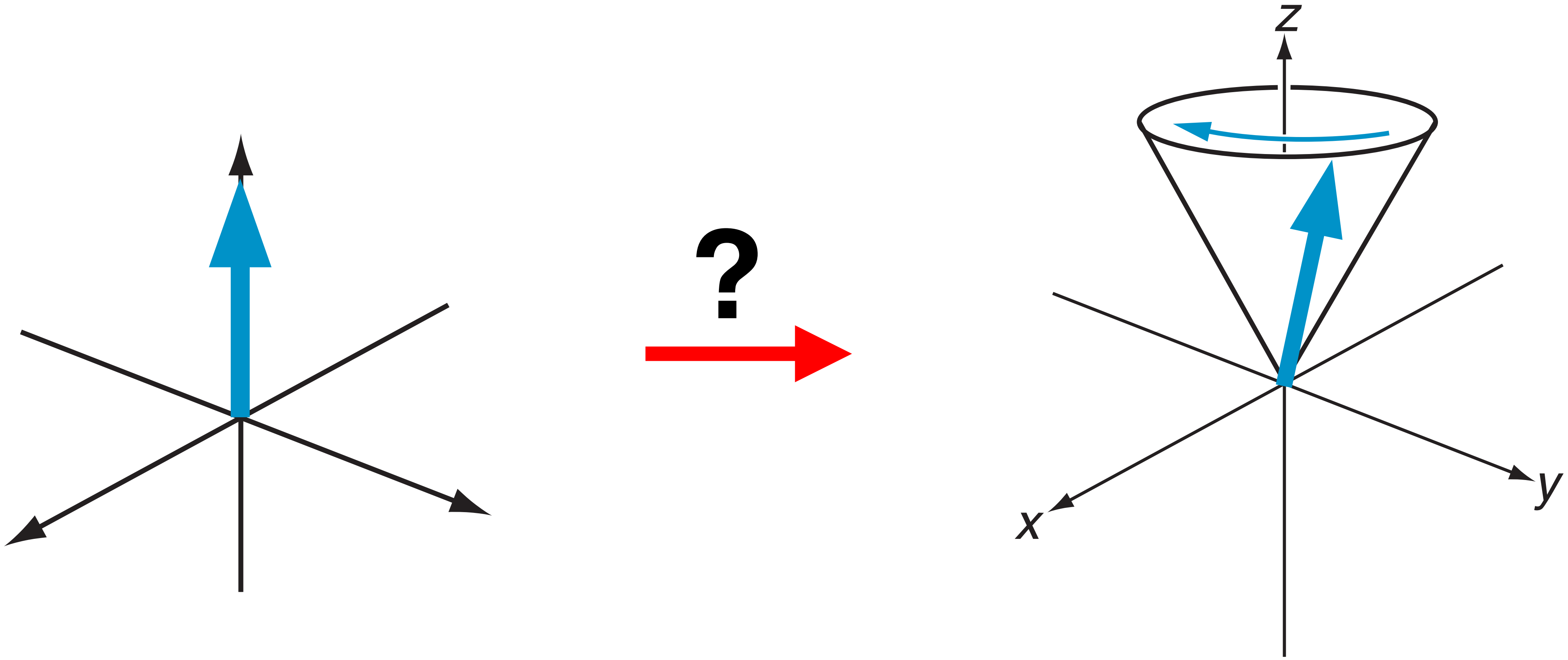
Time Evolution of the Transverse Magnetization



At time zero the magnetization is positioned such that its x-component is $M_0 \sin\beta$ and its y-component is zero.

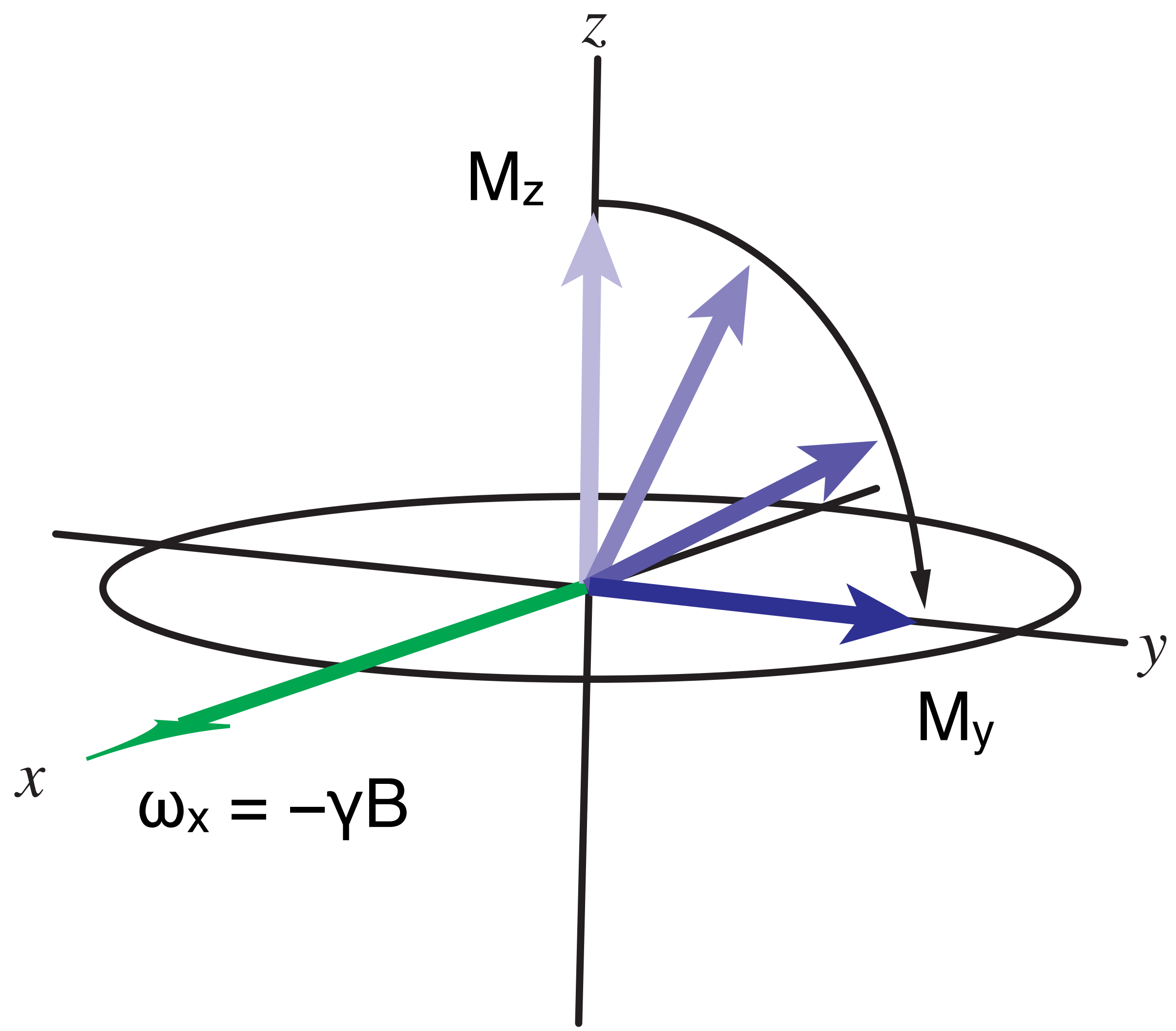
After time t, the angle through which the vector has rotated is $\omega_0 t$, so the x- and y-components of the magnetization are $M_0 \sin\beta \cos\omega_0 t$ and $-M_0 \sin\beta \sin\omega_0 t$.

How can we tip the magnetization out of equilibrium?



Perturbing the Magnetisation

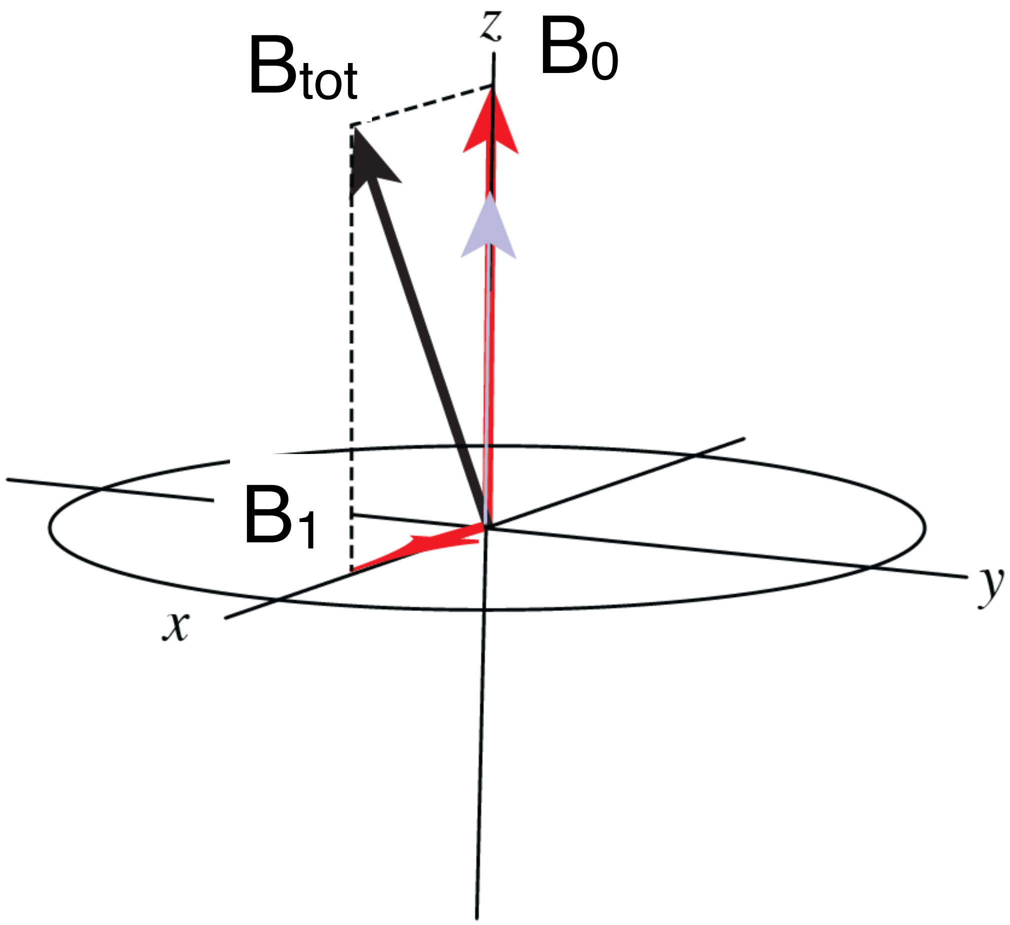
1. Brute force & sudden change



A sudden change in the direction of the magnetic field could induce a perturbation of the magnetisation… but sudden is not practical.

Perturbing the Magnetisation

1a. Adding an Extra Field

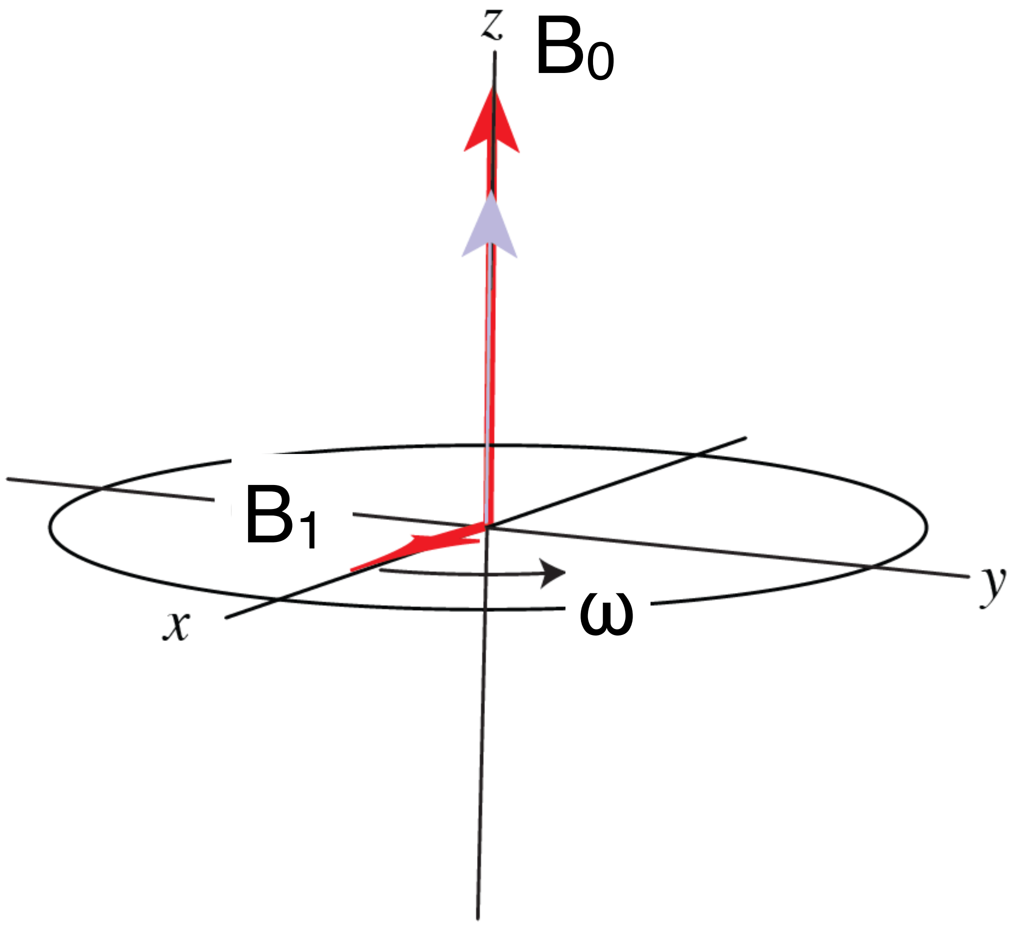


Adding a second component to the field in a different direction could also in principle perturb the magnetisation…
In practice, $\gamma B_1 < 100$ kHz (remember γB_0 is typically 500 MHz)

Perturbing the Magnetisation

Magnetic Resonance

$$B_1 = B_0 - B_{1x}\mathit{cos}\omega t + B_{1y}\sin\omega t$$

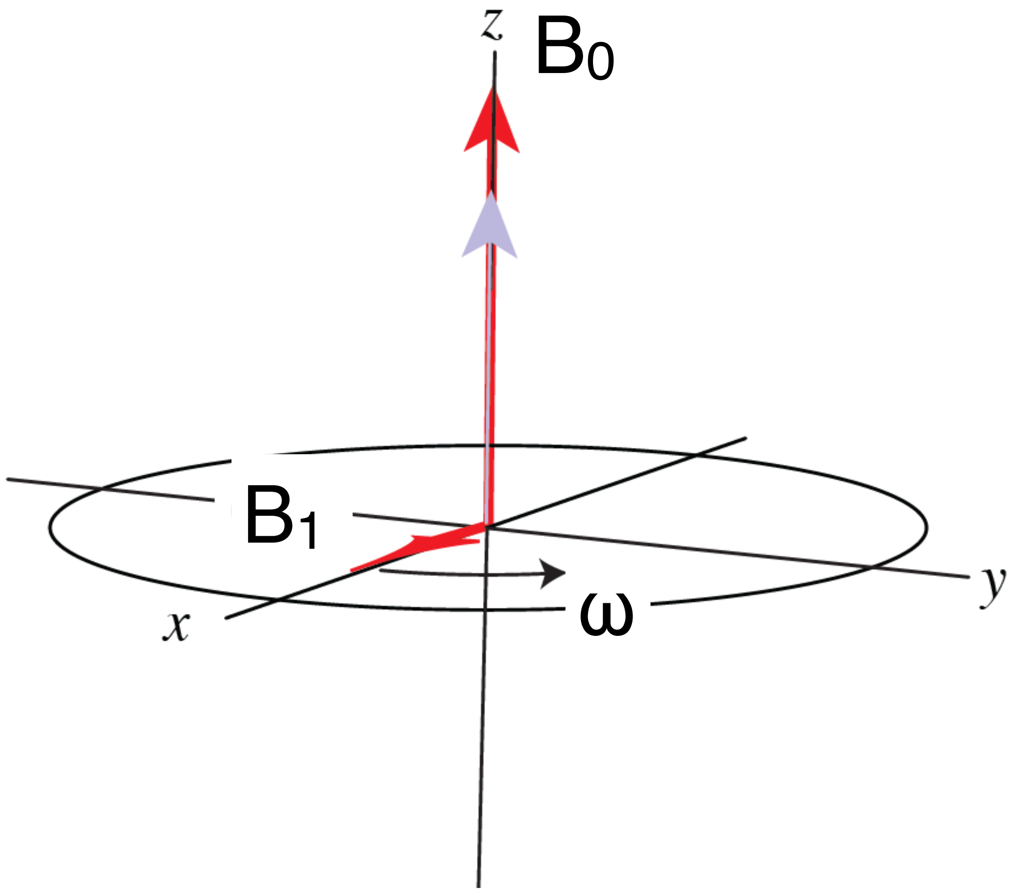


Adding a small oscillating field B_1 perpendicular to B_0 can cause resonance effects: even for $B_1 \ll B_0$

Perturbing the Magnetisation

Magnetic Resonance

$$B_1 = B_0 - B_{1x} \mathit{cos}\omega t + B_{1y}\sin\omega t$$

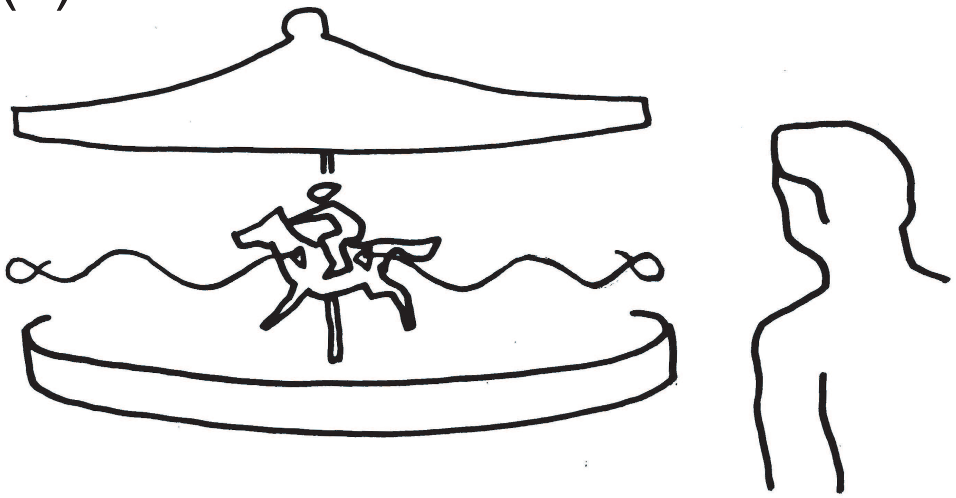


To study this **time dependent interaction**, the most enlightening method is to remove the time dependence by transforming to a **rotating frame**.

This can be understood geometrically (visually) or mathematically.

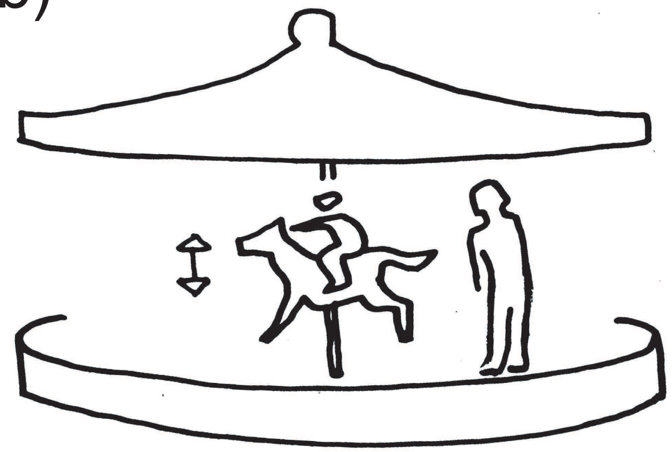
The Rotating Frame

(a)



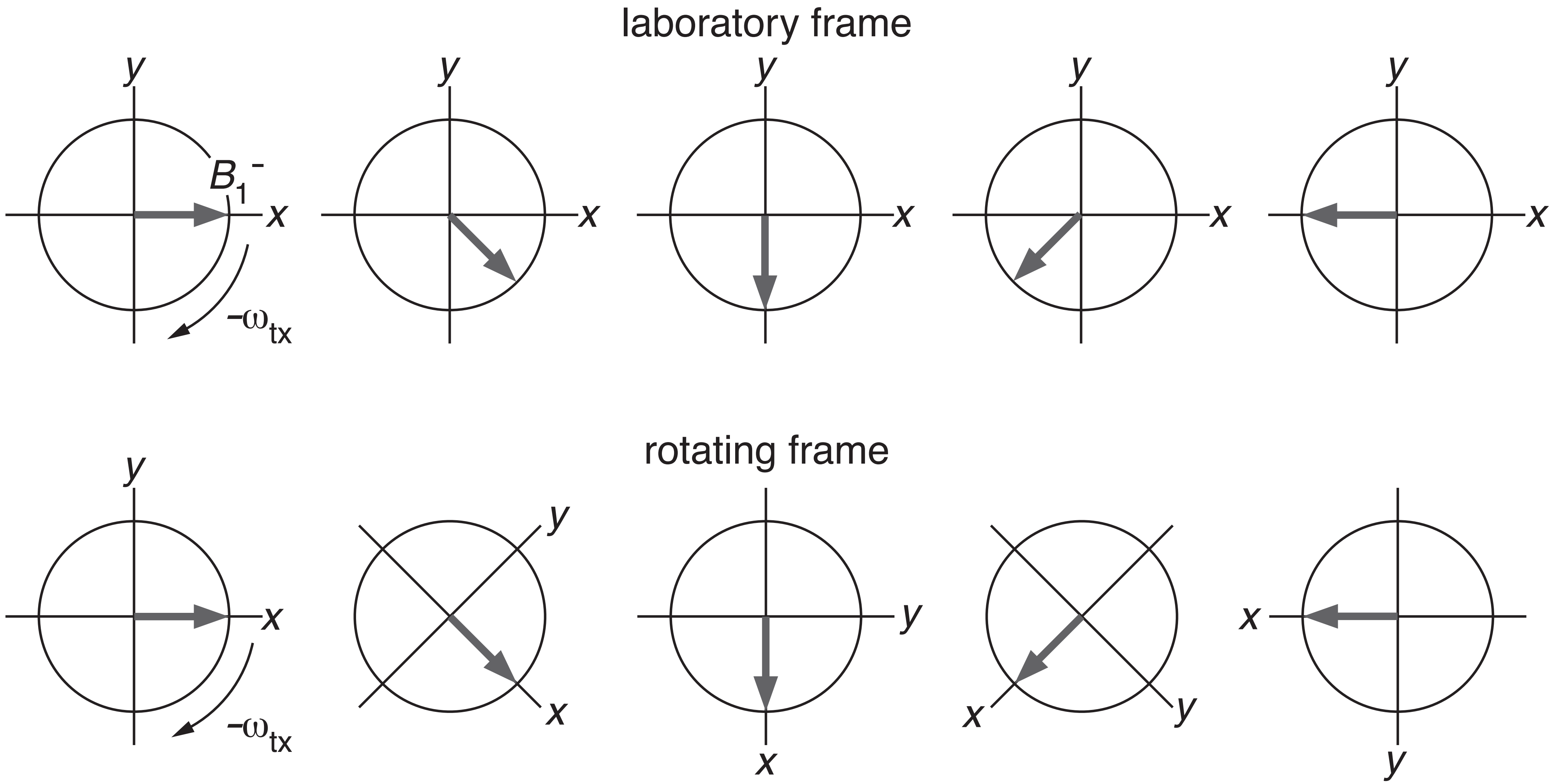
A child riding on a merry-go-round executes a complex motion as seen by a fixed observer.

(b)



If the observer stands on the merry-go-round the child appears to be executing a simple up–down motion.

The Rotating Frame



In a reference frame rotating at the Larmor frequency, transverse magnetisation appears static (here fixed on the x axis).

(Of course the precession has not really stopped, it is just that we are viewing it differently.)

The Reduced Field in the Rotating Frame

If the rotating frame frequency is $\omega_{rot.\ frame}$, the ***apparent (or effective) frequency*** of the Larmor precession in such a frame will be $(\omega_0 - \omega_{rot.\ frame})$.

This difference frequency is called the ***offset***, and is given the symbol Ω:

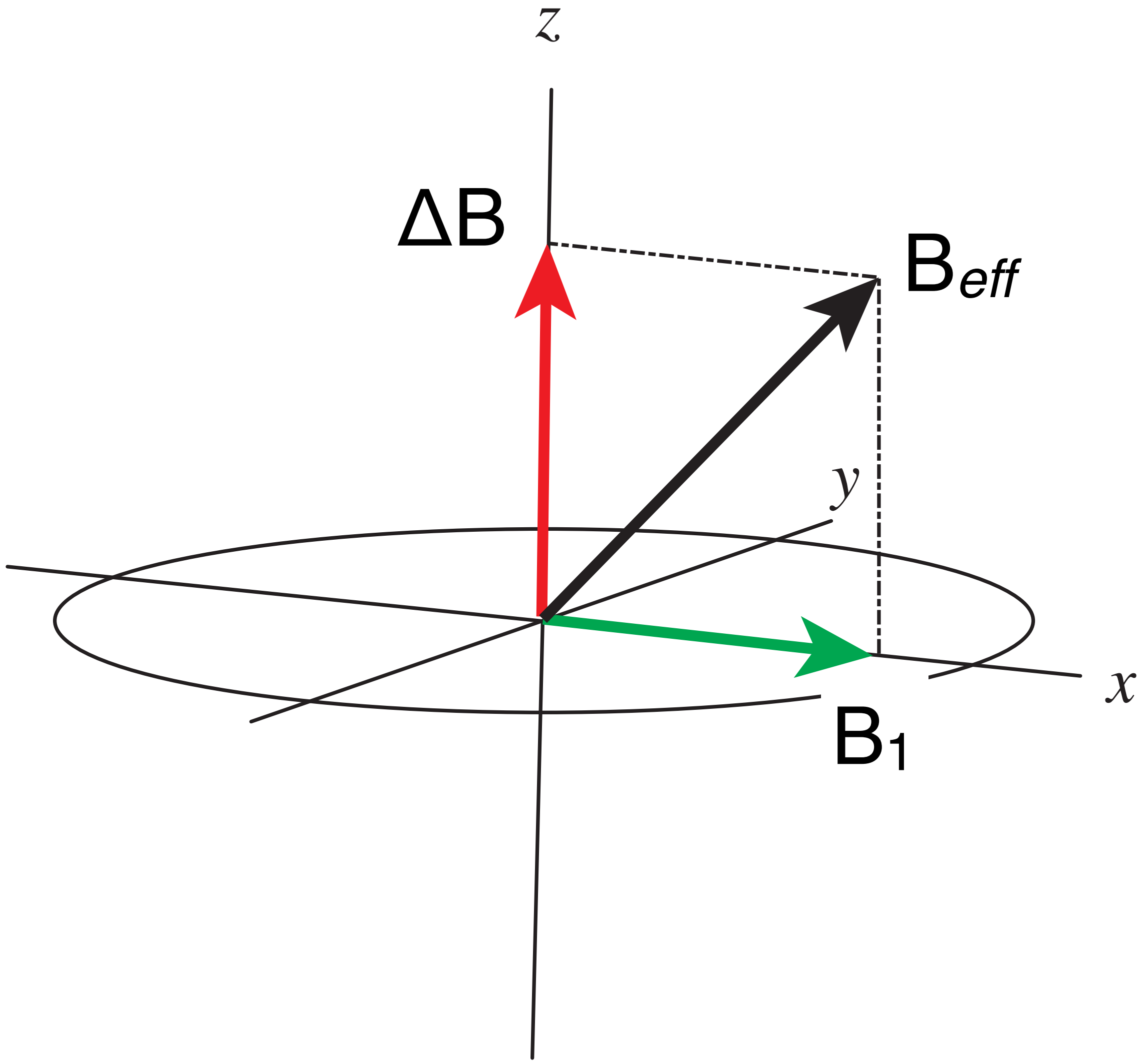
$\Omega = \omega_0 - \omega_{\mathrm{rot.\ frame}}$

Following this line of argument we can say that if in the rotating frame the precessional frequency appears to be Ω, then the apparent magnetic field ΔB is given by

$\Delta B = -\Omega/\gamma$.

ΔB is called the ***reduced field*** in the rotating frame. Clearly if the offset is zero, then so too is the reduced field.

Effective Fields in the Rotating Frame



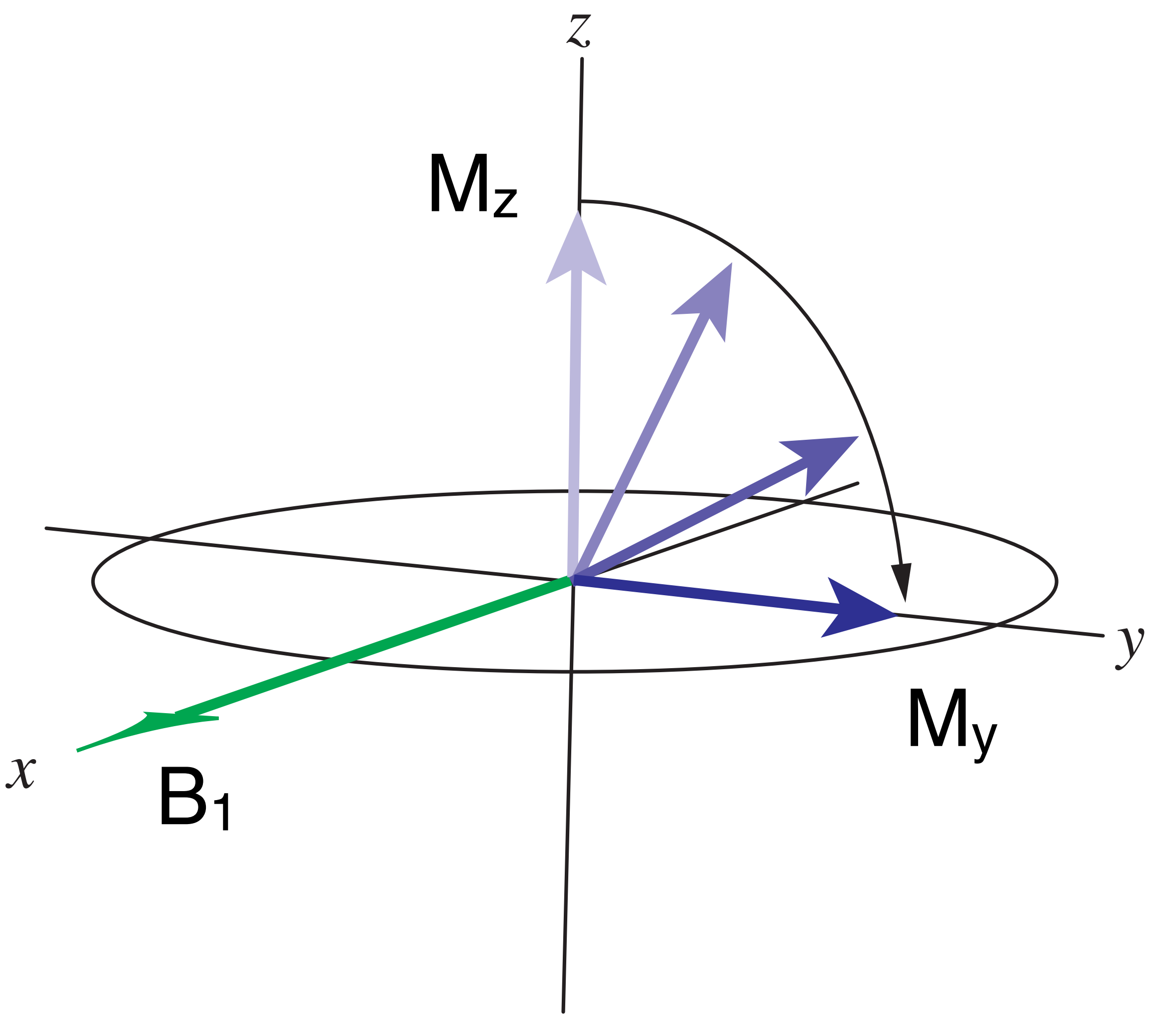
If the offset is zero, then so too is the reduced field ΔB. More generally if the oscillation frequency of the B_1 field $\omega_{\text{rot. frame}}$ is chosen to be close to the Larmor frequency ω_0 (such that ΔB is comparable to or smaller than B_1), then the effective field can be significantly tilted away from the z axes in the rotating frame.

Resonant Pulses

When $\Delta B = 0$ the B_1 irradiation is said to be “on resonance” and the effective field is purely transverse. The magnetisation precesses around the B_1 field (here applied along the *x* axis of the rotating frame) with frequency $\omega_1 = -\gamma B_1$.

If the irradiation is applied for a time τ such that $\omega_1\tau = \pi/2$, M_z will be converted into pure M_y, for any value of ω_1.

We refer to this as a π/2 (or 90°) pulse



Resonant Pulses (Callaghan)

https://www.youtube.com/watch?v=7aRKAXD4dAg

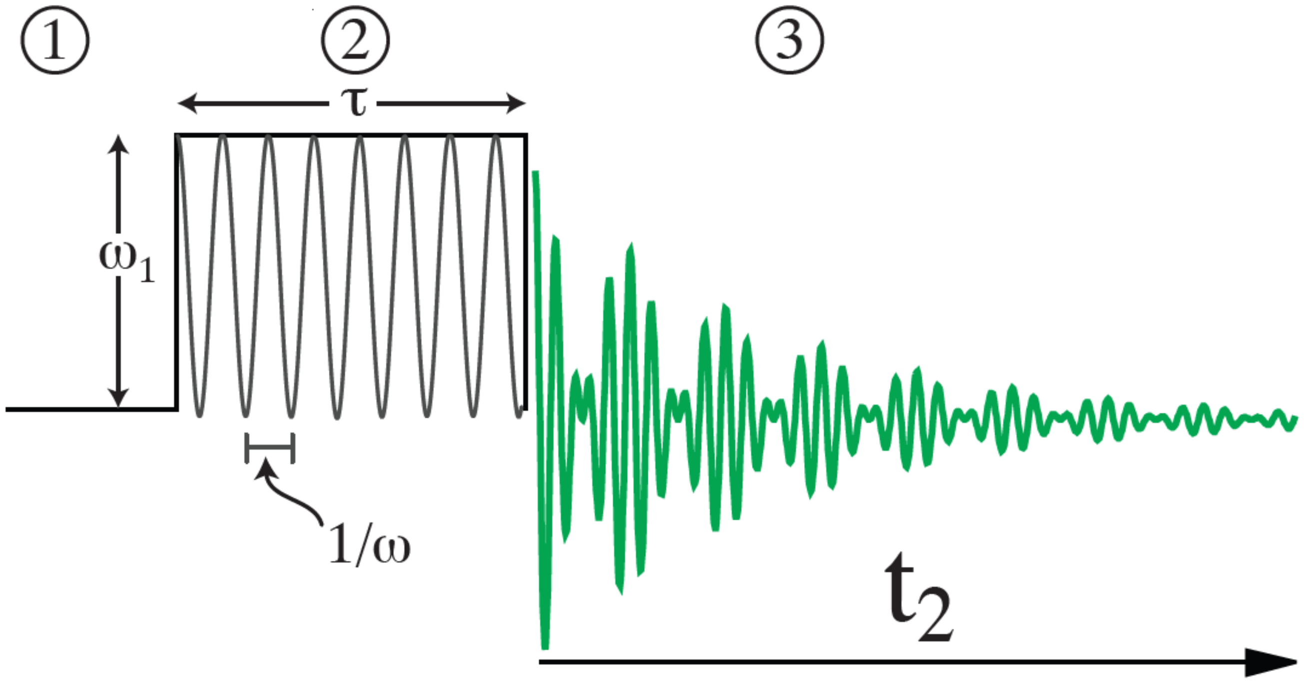
Pulsed FTNMR Spectroscopy

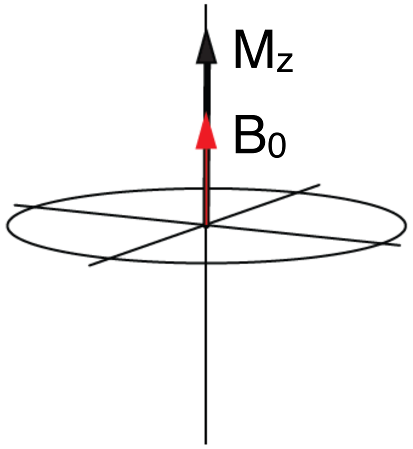
$\omega_1 = -\gamma B_1$

$\omega_1 \tau = \pi/2$

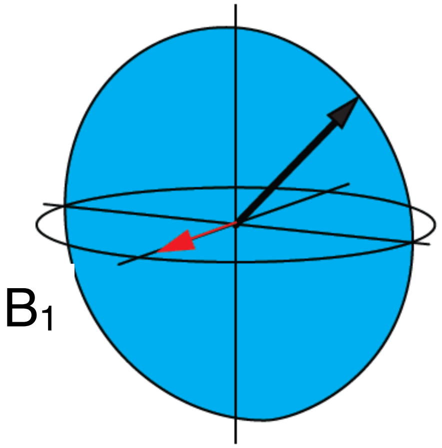
ω = carrier frequency,
chosen by the operator to be
near to the resonance frequencies

^{1}H

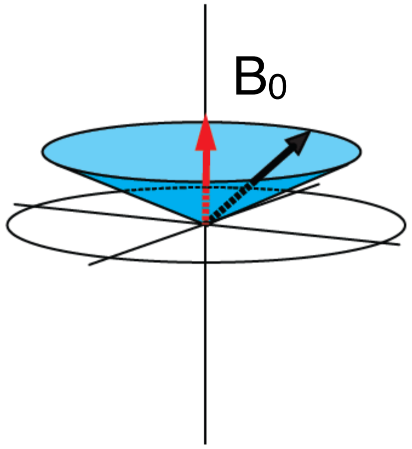




1. Equilibrium. The net magnetization is aligned along the direction of the main field (z-axis).



2. A field is applied in the transverse plane. The magnetization of the ensemble precesses around the field.



3. The field is removed leaving a net transverse component of the ensemble magnetization. This ***coherence*** then starts to precess around the main field.

What Happens After a Pulse? Relaxation

After a pulse, the system will oscillate under Larmor precession, but it must inevitably return to equilibrium.

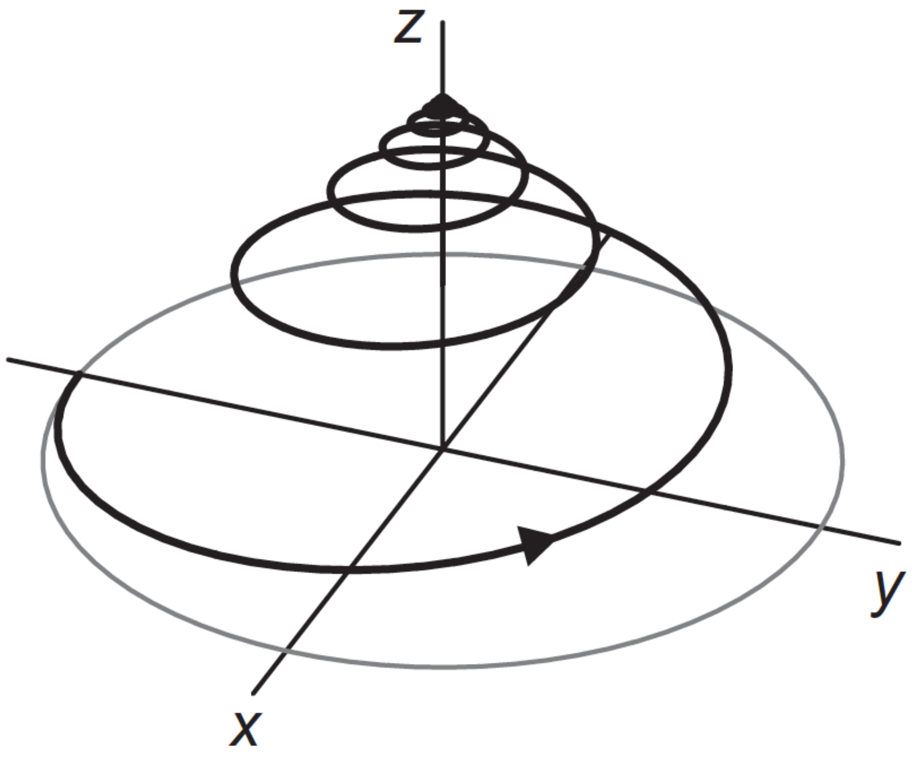
The longitudinal component (i.e. that aligned with the field (here z)) relaxes towards the equilibrium value (a non-zero value) M_0.

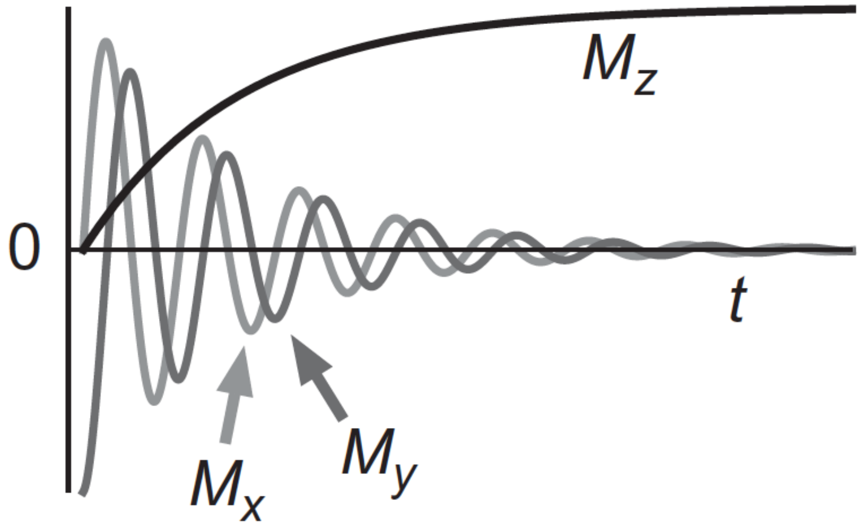
The transverse components both relax back to zero.

Evolution of M_z towards equilibrium modifies the total energy of the system: there must be exchange of energy with the lattice.

Relaxation of the transverse components does not modify the total energy.

We thus distinguish ***longitudinal relaxation*** (or spin-lattice relaxation), T_1, from ***transverse relaxation*** (or spin-spin relaxation) relaxation, T_2.





The Bloch Equations

The relaxation equations are therefore:

$$\frac{d}{dt}M_z = -\frac{1}{T_1}(M_z - M_0)$$

$$\frac{d}{dt}M_{x,y} = -\frac{1}{T_2}M_{x,y}$$

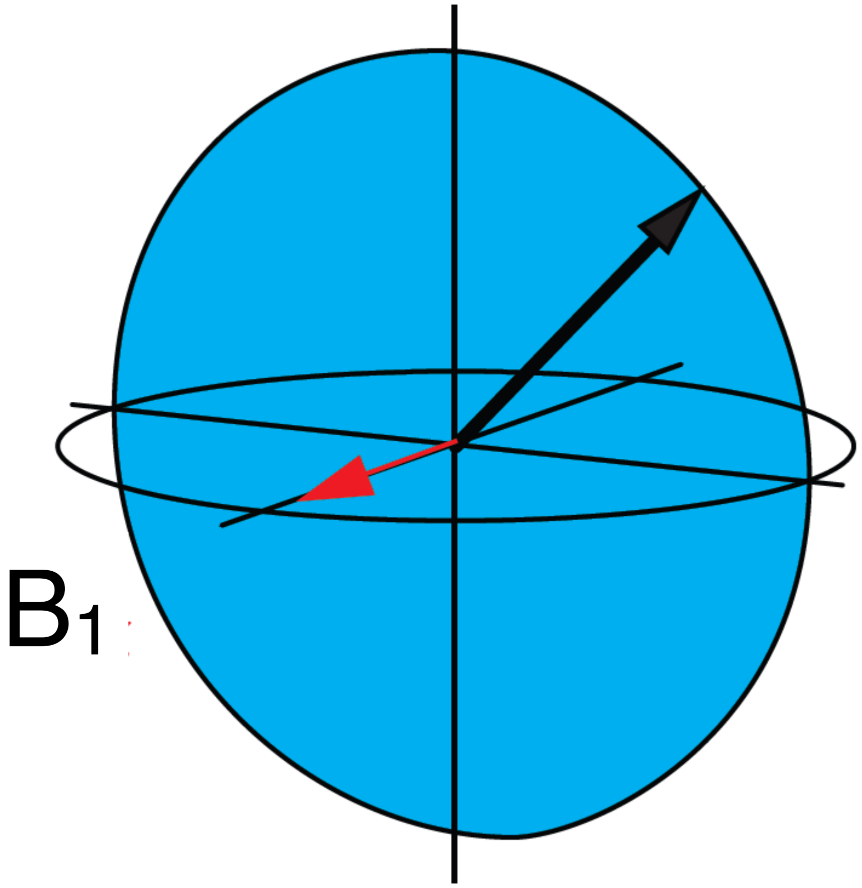
In usual NMR experiments, one uses in addition to the main field B0 much smaller time dependent fields that do not significantly alter the direction and size of the total field, and therefore does not change the relaxation rate T_1 nor (usually) T_2.

By adding the relaxation terms to the evolution due to Larmor precession one obtains the Bloch equations:

$$\frac{d}{dt}\boldsymbol{M} = \gamma\boldsymbol{M}\times\boldsymbol{B} - \frac{1}{T_1}(M_z - M_0)\boldsymbol{k} - \frac{1}{T_2}\left(M_x\boldsymbol{i} + M_y\boldsymbol{j}\right)$$

where ***i***, ***j*** and ***k*** are unit vectors in the *x, y* and *z* directions respectively.

Dances with Spins



Spin Inversion: $\omega_1\tau = \pi$

(*size of signal? state of polarisation?*)

"Self-Induced Transparency" $\omega_1\tau = 2\pi$

Conclusions

- Due to the slight preference of nuclear magnetic moments to align with a magnetic field, placing a sample in a field induces a very small bulk magnetism of the sample.
- This bulk magnetic moment can be described by a magnetisation vector.
- At equilibrium the magnetisation vector is aligned with the field.
- Resonant oscillating fields applied in a direction perpendicular to the main field can induce motion of the magnetisation vector. Pure transverse magnetisation can be created by an on-resonance pulse with $\omega_1\tau = \pi/2$.
- The magnetisation precesses around the magnetic field at the Larmor frequency. This precession induces an oscillating voltage in a detection coil.
- The longitudinal and transverse components of the magnetisation will relax back to equilibrium with rates T_1 and T_2 respectively. The Bloch equations provide a phenomenological equation of the motion of the magnetisation.

Homework

(Each group should jointly finalise and upload the answers to the jigsaws.)

before next week's class:

read Chapter 3 of
P.J. Hore, “Nuclear Magnetic Resonance,” 2nd Edition,
(Oxford University Press, 2015)

A pdf file of the Chapter is available on the course moodle page.

You do not need to read the greyed out parts
(but you can if you want to)